EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 4, 2012, 540-553
ISSN 1307-5543 – www.ejpam.com

Random Stability of a Functional Equation Related to an Inner Product Space

Dong Yun Shin[1], Jung Rye Lee[2,*], Choonkil Park[3]

[1] *Department of Mathematics, University of Seoul, Seoul 130-743, Korea*
[2] *Department of Mathematics, Daejin University, Kyeonggi 487-711, Korea*
[3] *Research Institute for Natural Sciences, Hanyang University, Seoul 133-791, Korea*

Abstract. In [14], Th.M. Rassias introduced the following equality

$$\sum_{i,j=1}^{n} \|x_i - x_j\|^2 = 2n \sum_{i=1}^{n} \|x_i\|^2, \qquad \sum_{i=1}^{n} x_i = 0$$

for a fixed integer $n \geq 3$. For a mapping $f : X \to Y$, where X is a vector space and Y is a complete random normed space, we consider the following functional equation

$$\sum_{i,j=1}^{n} f(x_i - x_j) = 2n \sum_{i=1}^{n} f(x_i) \tag{1}$$

for all $x_1, \ldots, x_n \in X$ with $\sum_{i=1}^{n} x_i = 0$. In this paper, we prove the Hyers-Ulam stability of the functional equation (1) related to an inner product space.

2010 Mathematics Subject Classifications: 39B52, 46S50, 46C05, 47S50, 26E50.
Key Words and Phrases: random normed space, Hyers-Ulam stability, quadratic functional equation, inner product space.

1. Introduction

A square norm on an inner product space satisfies the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

From the above equation, we consider the following functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

*Corresponding author.

Email addresses: `dyshin@uos.ac.kr` (D. Shin), `jrlee@daejin.ac.kr` (J. Lee), `baak@hanyang.ac.kr` (C. Park)

related to an inner product space. The stability problem of functional equations originated from a question of S.M. Ulam [18] concerning the stability of group homomorphisms. D.H. Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces and Hyers' Theorem was generalized by Th.M. Rassias [13] for linear mappings by considering an unbounded Cauchy difference. Especially, the Hyers-Ulam stability of the above functional equation related to an inner product space has been studied [see 7, 17].

A square norm on an inner product space also satisfies

$$\sum_{i,j=1}^{3} \|x_i - x_j\|^2 = 6\sum_{i=1}^{3} \|x_i\|^2$$

for all $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 + x_2 + x_3 = 0$ [see 14]. From the above equality we can define the functional equation

$$f(x-y) + f(2x+y) + f(x+2y) = 3f(x) + 3f(y) + 3f(x+y),$$

which is called a *quadratic functional equation*. In fact, $f(x) = ax^2$ in $\mathbb{R}$ satisfies the quadratic functional equation.

The aim of this paper is to investigate the Hyers-Ulam stability of additive-quadratic functional equation in a random normed space related to an inner product space.

Throughout this paper, we use the definition of a random normed space as in [1, 10, 15, 16]. Δ^+ is the space of distribution functions that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ which is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions F for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \le 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1 ([15]). *A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:*

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Recall that if T is a t-norm and $\{x_n\}$ is a sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \ge 2$ [see 3]. $T_{i=1}^{\infty} x_i$ is defined as $\lim_{m\to\infty} T_{i=1}^{m} x_i$.

Definition 2 ([16]). *A* random normed space *(briefly, RN-space) is a triple* (X,μ,T)*, where* X *is a vector space,* T *is a continuous* t*-norm and* μ *is a mapping from* X *into* D^+ *satisfies the following conditions:*

(RN1) $\mu_x(t)=\varepsilon_0(t)$ *for all* $t>0$ *if and only if* $x=0$*;*

(RN2) $\mu_{\alpha x}(t)=\mu_x(\frac{t}{|\alpha|})$ *for all* $x\in X$*,* $\alpha\neq 0$*;*

(RN3) $\mu_{x+y}(t+s)\geq T(\mu_x(t),\mu_y(s))$ *for all* $x,y\in X$ *and* $t,s\geq 0$*.*

A sequence $\{x_n\}$ in an RN-space (X,μ,T) is said to be *convergent* to x in X if, for every $\epsilon>0$ and $\lambda>0$, there exists a positive integer N such that $\mu_{x_n-x}(\epsilon)>1-\lambda$ whenever $n\geq N$. An RN-space (X,μ,T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

The Hyers-Ulam stability of functional equations in random normed spaces and fuzzy normed spaces has been studied [see 3, 4, 6, 8, 9, 11, 12]. Let V,W be vector spaces. It is shown that if a mapping $f:V\to W$ satisfies the functional equation (1), then the mapping f is the sum of an additive mapping and a quadratic mapping [see 2]. In this paper, we investigate the Hyers-Ulam stability of the functional equation (1) in RN-spaces.

Throughout this paper, assume that X is a vector space and that (Y,μ,T) is a complete RN-space.

2. Hyers-Ulam Stability of the Functional Equation (1): An Odd Case

We investigate the functional equation (1) for an odd mapping in RN-spaces.

For a given mapping $f:X\to Y$, we define

$$Df(x_1,\ldots,x_n):=\sum_{i,j=1}^{n}f(x_i-x_j)-2n\sum_{i=1}^{n}f(x_i)$$

for all $x_1,\ldots,x_n\in X$ with $\sum_{i=1}^n x_i=0$.

For an odd mapping $f:X\to Y$, we note that if f satisfies

$$Df(x_1,x_2,\ldots,x_n)=0$$

for all $x_1,\ldots,x_n\in X$ with $\sum_{i=1}^n x_i=0$ then the mapping f is additive.

We prove the Hyers-Ulam stability of the functional equation (1) of an odd mapping in RN-spaces.

Theorem 1. *Let* $f:X\to Y$ *be an odd mapping for which there is a* $\rho:X^n\to D^+$ *(* $\rho(x_1,x_2,\ldots,x_n)$ *is denoted by* $\rho_{(x_1,x_2,\ldots,x_n)}$*) such that*

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t)\geq\rho_{(x_1,x_2,\ldots,x_n)}(t) \tag{2}$$

for all $(x_1,x_2,\ldots,x_n)\in X^n$ *and all* $t>0$*. If*

$$T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^{k+l}},\frac{x}{2^{k+l}},-\frac{x}{2^{k+l-1}},0,\ldots,0\right)}\left(\frac{nt}{2^{2k+l-2}}\right)=1 \tag{3}$$

and

$$\lim_{m\to\infty}\rho_{\left(\frac{x}{2^m},\frac{y}{2^m},-\frac{x+y}{2^m},0,\dots,0\right)}\left(\frac{nt}{2^{m-1}}\right)=1 \tag{4}$$

for all $x,y\in X$, all $t>0$ and all $l=0,1,2,\dots$, then there exists a unique additive mapping $A:X\to Y$ such that

$$\mu_{f(x)-A(x)}(t)\geq T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right) \tag{5}$$

for all $x\in X$ and all $t>0$.

Proof. Putting $x_1=x_2=\frac{x}{2},x_3=-x,x_4=\dots=x_n=0$ in (2), we get

$$\mu_{2n\left(f(x)-2f\left(\frac{x}{2}\right)\right)}(t)\geq\rho_{\left(\frac{x}{2},\frac{x}{2},-x,0,\dots,0\right)}(t)$$

which is equivalent to

$$\mu_{f(x)-2f\left(\frac{x}{2}\right)}(t)\geq\rho_{\left(\frac{x}{2},\frac{x}{2},-x,0,\dots,0\right)}(2nt)$$

for all $x\in X$ and all $t>0$. Replacing x and t by $\frac{x}{2^{k-1}}$ and $\frac{t}{2^{2k-1}}$, respectively in the above inequality, we get

$$\mu_{2^{k-1}f\left(\frac{x}{2^{k-1}}\right)-2^kf\left(\frac{x}{2^k}\right)}\left(\frac{t}{2^k}\right)\geq\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right)$$

for all $x\in X$ and all $t>0$.

Since $\mu_x(s)\leq\mu_x(t)$ for all s and t with $0<s\leq t$, we obtain

$$\begin{aligned}\mu_{f(x)-2^mf\left(\frac{x}{2^m}\right)}(t)=&\mu_{\sum_{k=1}^m\left(2^{k-1}f\left(\frac{x}{2^{k-1}}\right)-2^kf\left(\frac{x}{2^k}\right)\right)}(t)\\ \geq&\mu_{\sum_{k=1}^m\left(2^{k-1}f\left(\frac{x}{2^{k-1}}\right)-2^kf\left(\frac{x}{2^k}\right)\right)}\left(\sum_{k=1}^m\frac{t}{2^k}\right)\\ \geq&T_{k=1}^m\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right)\end{aligned}$$

Replacing x by $\frac{x}{2^l}$ in the above inequality, we get

$$\mu_{f\left(\frac{x}{2^l}\right)-2^mf\left(\frac{x}{2^{m+l}}\right)}(t)\geq T_{k=1}^m\rho_{\left(\frac{x}{2^{k+l}},\frac{x}{2^{k+l}},-\frac{x}{2^{k+l-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right)$$

which is equivalent to

$$\mu_{2^lf\left(\frac{x}{2^l}\right)-2^{m+l}f\left(\frac{x}{2^{m+l}}\right)}(t)\geq T_{k=1}^m\rho_{\left(\frac{x}{2^{k+l}},\frac{x}{2^{k+l}},-\frac{x}{2^{k+l-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k+l-2}}\right) \tag{6}$$

for all $x\in X$, all $t>0$ and all $l=0,1,2,\dots$.

Since the right hand side of the inequality (6) tends to 1 as $m \to \infty$ by (3), the sequence $\{2^m f\left(\frac{x}{2^m}\right)\}$ is a Cauchy sequence. Thus we define $A(x) := \lim_{m\to\infty} 2^m f\left(\frac{x}{2^m}\right)$ for all $x \in X$, which is an odd mapping.

Now we show that A is an additive mapping. By (2), we get

$$\mu_{2^m\left(f\left(\frac{x+y}{2^m}\right)-f\left(\frac{x}{2^m}\right)-f\left(\frac{y}{2^m}\right)\right)}(t) \geq \rho_{\left(\frac{x}{2^m},\frac{y}{2^m},-\left(\frac{x+y}{2^m}\right),0,\ldots,0\right)}\left(\frac{nt}{2^{m-1}}\right).$$

Taking the limit as $m \to \infty$ in the above inequality, by (4), the mapping A is additive. By letting $l = 0$ and taking the limit as $m \to \infty$ in (6), we get (5).

Finally, to prove the uniqueness of the additive mapping A subject to (5), let us assume that there exists another additive mapping B which satisfies (5). Since

$$\begin{aligned}\mu_{A(x)-B(x)}(2t) =& \mu_{A(x)-2^m f\left(\frac{x}{2^m}\right)+2^m f\left(\frac{x}{2^m}\right)-B(x)}(2t)\\ \geq& T\left(\mu_{A(x)-2^m f\left(\frac{x}{2^m}\right)}(t), \mu_{2^m f\left(\frac{x}{2^m}\right)-B(x)}(t)\right)\end{aligned}$$

and

$$\lim_{m\to\infty} \mu_{A(x)-2^m f\left(\frac{x}{2^m}\right)} = \lim_{m\to\infty} \mu_{B(x)-2^m f\left(\frac{x}{2^m}\right)} = 1$$

for all $x \in X$ and all $t > 0$, we get

$$\lim_{m\to\infty} T\left(\mu_{A(x)-2^m f\left(\frac{x}{2^m}\right)}(t), \mu_{2^m f\left(\frac{x}{2^m}\right)-B(x)}(t)\right) = 1.$$

Thus we have $A = B$.

Corollary 1. *Let $\theta \geq 0$ and let p be a constant with $p > 1$. For a normed vector space X and complete RN-space Y, let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \frac{t}{t + \theta \sum_{i=1}^n ||x_i||^p}$$

for all $(x_1, x_2, \ldots, x_n) \in X$ with $\sum_{i=1}^n x_i = 0$ and all $t > 0$. If

$$T_{k=1}^{\infty}\left(\frac{2^{(k+l)p} nt}{2^{(k+l)p} nt + 2^{2k+l-2}(2 + 2^p)\theta ||x||^p}\right) = 1$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq T_{k=1}^{\infty}\left(\frac{2^{kp} nt}{2^{kp} nt + 2^{2k-2}(2 + 2^p)\theta ||x||^p}\right)$$

for all $x \in X$ and all $t > 0$.

Proof. If we define

$$\rho_{(x_1,x_2,\ldots,x_n)}(t)=\frac{t}{t+\theta\sum_{i=1}^{n}||x_i||^p}$$

and apply Theorem 1, then we get the desired result.

Theorem 2. *Let $f:X\to Y$ be an odd mapping for which there is a $\rho:X^n\to D^+$ satisfying* (2). *If*

$$T_{k=1}^{\infty}\rho_{\left(2^{k+l-2}x,2^{k+l-2}x,-2^{k+l-1}x,0,\ldots,0\right)}\left(2^{l+1}nt\right)=1 \tag{7}$$

and

$$\lim_{m\to\infty}\rho_{\left(2^m x,2^m y,-2^m(x+y),0,\ldots,0\right)}\left(2^{m+1}nt\right)=1 \tag{8}$$

for all $x,y\in X$, all $t>0$ and all $l=0,1,2,\ldots$, then there exists a unique additive mapping $A:X\to Y$ such that

$$\mu_{f(x)-A(x)}(t)\geq T_{k=1}^{\infty}\rho_{\left(2^{k-2}x,2^{k-2}x,-2^{k-1}x,0,\ldots,0\right)}(2nt) \tag{9}$$

for all $x\in X$ and all $t>0$.

Proof. Putting $x_1=x_2=x, x_3=-2x, x_4=\ldots=x_n=0$ in (2), we get

$$\mu_{2n\left(f(2x)-2f(x)\right)}(t)\geq\rho_{(x,x,-2x,0,\ldots,0)}(t)$$

which is equivalent to

$$\mu_{f(x)-\frac{1}{2}f(2x)}(t)\geq\rho_{\left(\frac{x}{2},\frac{x}{2},-x,0,\ldots,0\right)}(4nt)$$

for all $x\in X$ and all $t>0$. Replacing x and t by $2^{k-1}x$ and $2t$, respectively, in the above inequality, we get

$$\mu_{\frac{1}{2^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{2^k}f\left(2^k x\right)}\left(\frac{t}{2^k}\right)\geq\rho_{\left(2^{k-2}x,2^{k-2}x,-2^{k-1}x,0,\ldots,0\right)}(2nt)$$

for all $x\in X$ and all $t>0$.

Since $\mu_x(s)\leq\mu_x(t)$ for all s and t with $0<s\leq t$, we obtain

$$\begin{aligned}\mu_{f(x)-\frac{1}{2^m}f(2^m x)}(t)=&\mu_{\sum_{k=1}^{m}\left(\frac{1}{2^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{2^k}f\left(2^k x\right)\right)}(t)\\ \geq&\mu_{\sum_{k=1}^{m}\left(\frac{1}{2^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{2^k}f\left(2^k x\right)\right)}\left(\sum_{k=1}^{m}\frac{t}{2^k}\right)\\ \geq&T_{k=1}^{m}\rho_{\left(2^{k-2}x,2^{k-2}x,-2^{k-1}x,0,\ldots,0\right)}(2nt)\end{aligned}$$

Replacing x by $2^l x$ in the above inequality, we get

$$\mu_{f\left(2^l x\right)-\frac{1}{2^m}f\left(2^{m+l}x\right)}(t)\geq T_{k=1}^{m}\rho_{\left(2^{k+l-2}x,2^{k+l-2}x,-2^{k+l-1}x,0,\ldots,0\right)}(2nt)$$

which is equivalent to

$$\mu_{\frac{1}{2^l}f(2^l x)-\frac{1}{2^{m+l}}f(2^{m+l}x)}(t) \geq T_{k=1}^{m}\rho_{(2^{k+l-2}x,2^{k+l-2}x,-2^{k+l-1}x,0,\ldots,0)}\left(2^{l+1}nt\right) \tag{10}$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$.

Since the right hand side of the inequality (10) tends to 1 as $m \to \infty$ by (7), the sequence $\{\frac{1}{2^m}f(2^m x)\}$ is a Cauchy sequence. Thus we define $A(x) := \lim_{m\to\infty}\frac{1}{2^m}f(2^m x)$ for all $x \in X$, which is an odd mapping.

Now we show that A is an additive mapping. By (2), we get

$$\mu_{\frac{1}{2^m}(f(2^m(x+y))-f(2^m x)-f(2^m y))}(t) \geq \rho_{(2^m x,2^m y,-2^m(x+y),0,\ldots,0)}(2^{m+1}nt).$$

Taking the limit as $m \to \infty$ in the above inequality, by (8) the mapping A is additive. By letting $l = 0$ an taking the limit as $m \to \infty$ in (10), we get (9).

The rest of the proof is the same as in the proof of Theorem 1.

Corollary 2. *Let $\theta \geq 0$ and let p be a constant with $0 < p < 1$. For a normed vector space X and complete RN-space Y, let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \frac{t}{t+\theta\sum_{i=1}^{n}||x_i||^p}$$

for all $(x_1, x_2, \ldots, x_n) \in X$ with $\sum_{i=1}^{n} x_i = 0$ and all $t > 0$. If

$$T_{k=1}^{\infty}\left(\frac{2^{l+1}nt}{2^{l+1}nt+2^{(k+l-1)p}(2^{1-p}+1)\theta||x||^p}\right) = 1$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x)-A(x)}(t) \geq T_{k=1}^{\infty}\left(\frac{2nt}{2nt+2^{(k-1)p}(2^{1-p}+1)\theta||x||^p}\right)$$

for all $x \in X$ and all $t > 0$.

Proof. If we define

$$\rho_{(x_1,x_2,\ldots,x_n)}(t) = \frac{t}{t+\theta\sum_{i=1}^{n}||x_i||^p}$$

and apply Theorem 2, then we get the desired result.

3. Hyers-Ulam Stability of the Functional Equation (1): An Even Case

We prove the Hyers-Ulam stability of the functional equation (1) of an even mapping in RN-spaces.

For an even mapping $f : X \to Y$ with $f(0) = 0$, we note that if f satisfies

$$Df(x_1, x_2, \ldots, x_n) = 0$$

for all $x_1, \ldots, x_n \in X$ with $\sum_{i=1}^{n} x_i = 0$ then the mapping f is quadratic.

Theorem 3. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$ for which there is a $\rho : X^n \to D^+$ satisfying* (2). *If*

$$T_{k=1}^{\infty} \rho_{\left(\frac{x}{2^{k+l}}, -\frac{x}{2^{k+l}}, 0, \ldots, 0\right)} \left(\frac{t}{2^{3k+2l-3}} \right) = 1 \tag{11}$$

and

$$\lim_{m\to\infty} \rho_{\left(\frac{x}{2^m}, \frac{y}{2^m}, -\frac{x+y}{2^m}, 0, \ldots, 0\right)} \left(\frac{t}{2^{2m-1}} \right) = 1 \tag{12}$$

for all $x, y \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{k=1}^{\infty} \rho_{\left(\frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0\right)} \left(\frac{t}{2^{3k-3}} \right) \tag{13}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $x_1 = x, x_2 = -x, x_3 = \ldots = x_n = 0$ in (2), we get

$$\mu_{2\left(f(2x)-4f(x)\right)}(t) \geq \rho_{(x,-x,0,\ldots,0)}(t)$$

which is equivalent to

$$\mu_{f(x)-4f\left(\frac{x}{2}\right)}(t) \geq \rho_{\left(\frac{x}{2}, -\frac{x}{2}, 0, \ldots, 0\right)}(2t)$$

for all $x \in X$ and all $t > 0$. Replacing x and t by $\frac{x}{2^{k-1}}$ and $\frac{t}{2^{3k-2}}$, respectively in the above inequality, we get

$$\mu_{4^{k-1}f\left(\frac{x}{2^{k-1}}\right)-4^k f\left(\frac{x}{2^k}\right)} \left(\frac{t}{2^k} \right) \geq \rho_{\left(\frac{x}{2^k}, -\frac{x}{2^k} 0, \ldots, 0\right)} \left(\frac{t}{2^{3k-3}} \right)$$

for all $x \in X$ and all $t > 0$.

Since $\mu_x(s) \leq \mu_x(t)$ for all s and t with $0 < s \leq t$, we obtain

$$\begin{aligned}
\mu_{f(x)-4^m f\left(\frac{x}{2^m}\right)}(t) =& \mu_{\sum_{k=1}^{m}\left(4^{k-1}f\left(\frac{x}{2^{k-1}}\right)-4^k f\left(\frac{x}{2^k}\right)\right)}(t) \\
\geq& \mu_{\sum_{k=1}^{m}\left(4^{k-1}f\left(\frac{x}{2^{k-1}}\right)-4^k f\left(\frac{x}{2^k}\right)\right)} \left(\sum_{k=1}^{m} \frac{t}{2^k} \right) \\
\geq& T_{k=1}^{m} \rho_{\left(\frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0\right)} \left(\frac{t}{2^{3k-3}} \right)
\end{aligned}$$

Replacing x by $\frac{x}{2^l}$ in the above inequality, we get

$$\mu_{f\left(\frac{x}{2^l}\right)-4^m f\left(\frac{x}{2^{m+l}}\right)}(t) \geq T_{k=1}^m \rho_{\left(\frac{x}{2^{k+l}},-\frac{x}{2^{k+l}},0,\dots,0\right)}\left(\frac{t}{2^{3k-3}}\right)$$

which is equivalent to

$$\mu_{4^l f\left(\frac{x}{2^l}\right)-4^{m+l} f\left(\frac{x}{2^{m+l}}\right)}(t) \geq T_{k=1}^m \rho_{\left(\frac{x}{2^{k+l}},-\frac{x}{2^{k+l}},0,\dots,0\right)}\left(\frac{t}{2^{3k+2l-3}}\right) \tag{14}$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \dots$.

Since the right hand side of the inequality (14) tends to 1 as $m \to \infty$ by (11), the sequence $\{4^m f\left(\frac{x}{2^m}\right)\}$ is a Cauchy sequence. Thus we define $Q(x) := \lim_{m\to\infty} 4^m f\left(\frac{x}{2^m}\right)$ for all $x \in X$, which is an even mapping.

Now we show that Q is an quadratic mapping. By (2), we get

$$\begin{aligned}&\mu_{4^m\left(f\left(\frac{x-y}{2^m}\right)+f\left(\frac{2x+y}{2^m}\right)+f\left(\frac{x+2y}{2^m}\right)-3f\left(\frac{x+y}{2^m}\right)-3f\left(\frac{x}{2^m}\right)-3f\left(\frac{y}{2^m}\right)\right)}(t)\\ &\geq \rho_{\left(\frac{x}{2^m},\frac{y}{2^m},-\frac{x+y}{2^m},0,\dots,0\right)}\left(\frac{t}{2^{2m-1}}\right).\end{aligned}$$

Taking the limit as $m \to \infty$ in the above inequality, by (12), the mapping Q is quadratic. Moreover, letting $l = 0$ and taking the limit as $m \to \infty$ in (14), we get (13).

The rest of the proof is the same as in the proof of Theorem 1.

Corollary 3. *Let $\theta \geq 0$ and let p be a constant with $p > 2$. For a normed vector space X and complete RN-space Y, let $f : X \to Y$ be an even mapping satisfying*

$$\mu_{Df(x_1,x_2,\dots,x_n)}(t) \geq \frac{t}{t + \theta \sum_{i=1}^n ||x_i||^p}$$

for all $(x_1, x_2, \dots, x_n) \in X$ with $\sum_{i=1}^n x_i = 0$ and all $t > 0$. If

$$T_{k=1}^{\infty}\left(\frac{2^{(k+l)p}t}{2^{(k+l)p}t + 2^{3k+2l-2}\theta||x||^p}\right) = 1$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \dots$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{k=1}^{\infty}\left(\frac{2^{kp}t}{2^{kp}t + 2^{3k-2}\theta||x||^p}\right)$$

for all $x \in X$ and all $t > 0$.

Proof. If we define

$$\rho_{(x_1,x_2,\dots,x_n)}(t) = \frac{t}{t + \theta \sum_{i=1}^n ||x_i||^p}$$

and apply Theorem 3, then we get the desired result.

Theorem 4. *Let $f : X \to Y$ be an even mapping with $f(0) = 0$ for which there is a $\rho : X^n \to D^+$ satisfying* (2). *If*

$$T_{k=1}^{\infty}\rho_{\left(2^{k+l-1}x,-2^{k+l-1}x,0,\dots,0\right)}\left(2^{k+2l-1}t\right) = 1 \tag{15}$$

and

$$\lim_{m\to\infty}\rho_{\left(2^m x,2^m y,-2^m(x+y),0,\dots,0\right)}\left(2^{m+1}t\right) = 1 \tag{16}$$

for all $x, y \in X$, all $t > 0$ and all $l = 0, 1, 2, \dots$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq T_{k=1}^{\infty}\rho_{\left(2^k x,-2^k x,0,\dots,0\right)}\left(2^{k-1}t\right) \tag{17}$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $x_1 = x, x_2 = -x, x_3 = \dots = x_n = 0$ in (2), we get

$$\mu_{2\left(f(2x)-4f(x)\right)}(t) \geq \rho_{(x,-x,0,\dots,0)}(t)$$

which is equivalent to

$$\mu_{f(x)-\frac{1}{4}f(2x)}\left(\frac{t}{4}\right) \geq \rho_{(x,-x,0,\dots,0)}(2t)$$

for all $x \in X$ and all $t > 0$. Replacing x and t by $2^{k-1}x$ and $2^{k-2}t$, respectively in the above inequality, we get

$$\mu_{\frac{1}{4^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{4^k}f\left(2^k x\right)}\left(\frac{t}{2^k}\right) \geq \rho_{\left(2^{k-1}x,-2^{k-1}x,0,\dots,0\right)}(2^{k-1}t)$$

for all $x \in X$ and all $t > 0$.

Since $\mu_x(s) \leq \mu_x(t)$ for all s and t with $0 < s \leq t$, we obtain

$$\begin{aligned}
\mu_{f(x)-\frac{1}{4^m}f(2^m x)}(t) =& \mu_{\sum_{k=1}^m\left(\frac{1}{4^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{4^k}f\left(2^k x\right)\right)}(t)\\
\geq& \mu_{\sum_{k=1}^m\left(\frac{1}{4^{k-1}}f\left(2^{k-1}x\right)-\frac{1}{4^k}f\left(2^k x\right)\right)}\left(\sum_{k=1}^m \frac{t}{2^k}\right)\\
\geq& T_{k=1}^m\rho_{\left(2^{k-1}x,-2^{k-1}x,0,\dots,0\right)}\left(2^{k-1}t\right)
\end{aligned}$$

Replacing x by $2^l x$ in the above inequality, we get

$$\mu_{f(2^l x)-\frac{1}{4^m}f\left(2^{m+l}x\right)}(t) \geq T_{k=1}^m\rho_{\left(2^{k+l-1}x,-2^{k+l-1}x,0,\dots,0\right)}\left(2^{k-1}t\right)$$

which is equivalent to

$$\mu_{\frac{1}{4^l}f(2^l x)-\frac{1}{4^{m+l}}f\left(2^{m+l}x\right)}(t) \geq T_{k=1}^m\rho_{\left(2^{k+l-1}x,-2^{k+l-1}x,0,\dots,0\right)}\left(2^{k+2l-1}t\right) \tag{18}$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \dots$.

Since the right hand side of the inequality (18) tends to 1 as $m \to \infty$ by (15), the sequence $\{\frac{1}{4^m} f(2^m x)\}$ is a Cauchy sequence. Thus we define $Q(x) := \lim_{m\to\infty} \frac{1}{4^m} f(2^m x)$ for all $x \in X$, which is an even mapping.

Now we show that Q is a quadratic mapping. By (2), we get

$$\mu_{\frac{1}{4^m}(f(2^m(x-y))+f(2^m(2x+y))+f(2^m(x+2y))-3f(2^m(x+y))-3f(2^m x)-3f(2^m y))}(t)$$
$$\geq \rho_{(2^m x, 2^m y, -2^m(x+y), 0, \ldots, 0)}(2^{m+1}t).$$

Taking the limit as $m \to \infty$ in the above inequality, by (16), the mapping Q is quadratic. Moreover, letting $l = 0$ and taking the limit as $m \to \infty$ in (18), we get (17).

The rest of the proof is the same as in the proof of Theorem 3.

Corollary 4. *Let $\theta \geq 0$ and let p be a constant with $0 < p < 2$. For a normed vector space X and complete RN-space Y, let $f : X \to Y$ be an even mapping satisfying*

$$\mu_{Df(x_1,x_2,\ldots,x_n)}(t) \geq \frac{t}{t + \theta \sum_{i=1}^n ||x_i||^p}$$

for all $(x_1, x_2, \ldots, x_n) \in X$ with $\sum_{i=1}^n x_i = 0$ and all $t > 0$. If

$$T_{k=1}^{\infty}\left(\frac{2^{k+2l-2}t}{2^{k+2l-2}t + 2^{(k+l)p}\theta||x||^p}\right) = 1$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \lim_{m\to\infty} T_{k=1}^{m}\left(\frac{2^{k-2}t}{2^{k-2}t + 2^{kp}\theta||x||^p}\right)$$

for all $x \in X$ and all $t > 0$.

Proof. If we define

$$\rho_{(x_1,x_2,\ldots,x_n)}(t) = \frac{t}{t + \theta \sum_{i=1}^n ||x_i||^p}$$

and apply Theorem 4, then we get the desired result.

4. Hyers-Ulam Stability of the Functional Equation (1)

We note that if a mapping $f : X \to Y$ satisfies the functional equation (1), then the mapping f is realized as the sum of an additive mapping and a quadratic mapping [see 2, Lemma 2.1].

Here, we let $g(x) := \frac{1}{2}(f(x) - f(-x))$ and $h(x) := \frac{1}{2}(f(x) + f(-x))$ for all $x \in X$. Then $g(x)$ is an odd mapping and $h(x)$ is an even mapping satisfying $f(x) = g(x) + h(x)$. Moreover, we get the following:

$$Dg(x_1, x_2, \ldots, x_n) = \frac{1}{2}\{Df(x_1, x_2, \ldots, x_n) - Df(-x_1, -x_2, \ldots, -x_n)\}$$

$$Dh(x_1,x_2,\dots,x_n)=\frac{1}{2}\{Df(x_1,x_2,\dots,x_n)+Df(-x_1,-x_2,\dots,-x_n)\}$$

for all $x_1,x_2,\dots,x_n\in X$.

Note that $Df(x_1,\dots,x_n)=0$ implies that $Dg(x_1,\dots,x_n)=0$ and $Dh(x_1,\dots,x_n)=0$.

Theorem 5. *Let $f:X\to Y$ be a mapping with $f(0)=0$ for which there is a $\rho:X^n\to D^+$ such that*

$$\mu_{Df(x_1,x_2,\dots,x_n)+Df(-x_1,-x_2,\dots,-x_n)}(2t)\geq\rho_{(x_1,x_2,\dots,x_n)}(t) \tag{19}$$

$$\mu_{Df(x_1,x_2,\dots,x_n)-Df(-x_1,-x_2,\dots,-x_n)}(2t)\geq\rho_{(x_1,x_2,\dots,x_n)}(t) \tag{20}$$

for all $(x_1,x_2,\dots,x_n)\in X^n$ and all $t>0$. If ρ satisfies (3)*,* (11) *and* (12)*, then there exist an additive mapping $A:X\to Y$ and a quadratic mapping $Q:X\to Y$ such that*

$$\begin{aligned}&\mu_{f(x)-A(x)-Q(x)}(2t)\\&\geq T\left(T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right),T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},-\frac{x}{2^k},0,\dots,0\right)}\left(\frac{t}{2^{3k-3}}\right)\right)\end{aligned}$$

for all $x\in X$ and all $t>0$.

Proof. Consider an odd mapping $g(x):=\frac{1}{2}(f(x)-f(-x))$ and an even mapping $h(x):=\frac{1}{2}(f(x)+f(-x))$ for all $x\in X$ with $f(x)=g(x)+h(x)$. By Theorem 1, there exists a unique additive mapping $A:X\to Y$ such that

$$\mu_{g(x)-A(x)}(t)\geq T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-3}}\right)$$

for all $x\in X$ and all $t>0$. And by Theorem 3, there exists a unique quadratic mapping $Q:X\to Y$ such that

$$\mu_{h(x)-Q(x)}(t)\geq T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},-\frac{x}{2^k},0,\dots,0\right)}\left(\frac{t}{2^{3k-3}}\right)$$

for all $x\in X$ and all $t>0$. Since $f(x)=g(x)+h(x)$, we obtain

$$\begin{aligned}\mu_{f(x)-A(x)-Q(x)}(2t)&=\mu_{g(x)-A(x)+h(x)-Q(x)}(2t)\\&\geq T(\mu_{g(x)-A(x)}(t),\mu_{h(x)-Q(x)}(t))\\&\geq T\left(T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},\frac{x}{2^k},-\frac{x}{2^{k-1}},0,\dots,0\right)}\left(\frac{nt}{2^{2k-2}}\right),T_{k=1}^{\infty}\rho_{\left(\frac{x}{2^k},-\frac{x}{2^k},0,\dots,0\right)}\left(\frac{t}{2^{3k-3}}\right)\right)\end{aligned}$$

for all $x\in X$ and all $t>0$, as desired.

Similarly, we can obtain the following. We will omit the proof.

Theorem 6. *Let $f:X\to Y$ be a mapping with $f(0)=0$ for which there is a $\rho:X^n\to D^+$ satisfying* (19) *and* (20)*. If ρ satisfies* (7)*,* (15) *and* (16)*, then there exist an additive mapping $A:X\to Y$ and a quadratic mapping $Q:X\to Y$ such that*

$$\begin{aligned}&\mu_{f(x)-A(x)-Q(x)}(2t)\\&\geq T\left(T_{k=1}^{\infty}\rho_{(2^{k-2}x,2^{k-2}x,-2^{k-1}x,0,\dots,0)}(2nt),T_{k=1}^{\infty}\rho_{(2^kx,-2^kx,0,\dots,0)}\left(2^{k-1}t\right)\right)\end{aligned}$$

for all $x\in X$ and all $t>0$.

ACKNOWLEDGEMENTS D. Y. Shin was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0021792).

References

[1] S.S. Chang, Y. Cho, and S. Kang. *Nonlinear Operator Theory in Probabilistic Metric Spaces.* Nova Science Publishers Inc., New York, 2001.

[2] Y. Cho, C. Park, and T.M. Rassias. Inner product spaces and functional equations. *Journal of Computational Analysis and Applications*, 13(2):296–304, 2011.

[3] M.E. Gordji, J.M. Rassias, and M.B. Savadkouhi. Approximation of the quadratic and cubic functional equation in RN-spaces. *European Journal of Pure and Applied Mathematics*, 2:494–507, 2009.

[4] M.E. Gordji, M.B. Savadkouhi, and J.M. Rassias. Stability of a mixed type additive and quadratic functional equation in random normed spaces. *Journal of Concrete and Applicable Mathematics*, 10(1-2):117–129, 2012.

[5] D.H. Hyers. On the stability of the linear functional equation. *Proceedings of the National Academy of Sciences of the United States of America*, 27:222–224, 1941.

[6] S. Jang and C. Park. Fuzzy stability of a functional equation related to inner product spaces. *Hacettepe Journal of Mathematics and Statistics*, 40(5):711–723, 2011.

[7] Pl. Kannappan. Quadratic functional equation and inner product spaces. *Results in Mathematics*, 27:368–372, 1995.

[8] D. Miheţ. The fixed point method for fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems*, 160:1663–1667, 2009.

[9] D. Miheţ. The probabilistic stability for a functional equation in a single variable. *Acta Mathematica Hungaria*, 123:249–256, 2009.

[10] D. Miheţ and V. Radu. On the stability of the additive Cauchy functional equation in random normed spaces. *Journal of Mathematical Analysis and Applications*, 343:567–572, 2008.

[11] A.K. Mirmostafaee, M. Mirzavaziri, and M.S. Moslehian. Fuzzy stability of the Jensen functional equation. *Fuzzy Sets and Systems*, 159:730–738, 2008.

[12] A.K. Mirmostafaee and M.S. Moslehian. Fuzzy approximately cubic mappings. *Information Sciences*, 178:3791–3798, 2008.

[13] T.M. Rassias. On the stability of the linear mapping in Banach spaces. *Proceedings of the American Mathematical Society*, 72:297–300, 1978.

[14] T.M. Rassias. On characterizations of inner product spaces and generalizations of the H. Bohr inequality. In T.M. Rassias et al., editor, *Topics in Mathematical Analysis.*, pages 803–819. World Scientific Publ. Co., Singapore, 1989.

[15] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* Elsevier, North Holand, New York, 1983.

[16] A.N. Sherstnev. On the notion of a random normed space. *Doklady Akademii Nauk SSSR*, 149:280–283, 1963.

[17] F. Skof. Propriet locali e approssimazione di operatori. *Rendiconti del Seminario Matematico e Fisico di Milano*, 53:113–129, 1983.

[18] S.M. Ulam. *Problems in Modern Mathematics.* Wiley, New York, 1960.