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Majorization for Certain Classes of Analytic Functions Defined by a Generalized Operator

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Abstract. In this paper, we investigate majorization properties for certain classes of multivalent analytic functions defined by a generalized operator. Also, we point out some new and known consequences of our main result.

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1. Introduction and preliminaries

Let $\mathscr{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_n z^{p+n}, \qquad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathscr{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathscr{A}_1 =: \mathscr{A}$. For functions $f_j \in \mathscr{A}_p$ given by

$$f_j(z) = z^p + \sum_{n=1}^{\infty} a_{n,j} z^{p+n}, \qquad (j = 1, 2; p \in \mathbb{N}), \tag{2}$$

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we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^{p+n} = (f_2 * f_1)(z).$$

Let $f(z)$ and $g(z)$ be analytic in $\mathscr{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathscr{U}$, if there exists an analytic function $w(z)$ in $\mathscr{U}$ with

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathscr{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathscr{U}).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathscr{U}$, then $f(z) \prec g(z) \quad (z \in \mathscr{U}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$.

Suppose that the functions $f(z)$ and $g(z)$ are analytic in the open unit disk $\mathscr{U}$. Then we say that the function $f(z)$ is majorized by $g(z)$ in $\mathscr{U}$ (see [5]) and write

$$f(z) \ll g(z) \qquad (z \in \mathscr{U}), \tag{3}$$

if there exists a function $\varphi(z)$, analytic in $\mathscr{U}$, such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \qquad (z \in \mathscr{U}).$$

The majorization (3) is closely related to the concept of quasi-subordination between analytic functions in $\mathscr{U}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \leq s+1)$ be complex numbers such that $\beta_l \neq 0, -1, -2, \ldots$ for $l \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \qquad (z \in \mathscr{U}),$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \text{ for } n \in \mathbb{N} \text{ and } (x)_0 = 1.$$

Corresponding to a function $\mathscr{G}^p_{q,s}(\alpha_1; \beta_1; z)$ defined by

$$\mathscr{G}_{q,s}(\alpha_1, \beta_1; z) := z^p \, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z), \tag{4}$$

C.Selvaraj and K.R.Karthikeyan recently defined the following generalized differential operator $D^{p,m}_\lambda(\alpha_1, \beta_1)f : \mathscr{A}_p \longrightarrow \mathscr{A}_p$ by

$$\begin{aligned}
&D^{p,0}_\lambda(\alpha_1, \beta_1)f(z) = f(z) * \mathscr{G}^p_{q,s}(\alpha_1, \beta_1; z),\\
&D^{p,1}_\lambda(\alpha_1, \beta_1)f(z) = (1-\lambda)(f(z) * \mathscr{G}^p_{q,s}(\alpha_1, \beta_1; z)) + \frac{\lambda}{p} z(f(z) * \mathscr{G}^p_{q,s}(\alpha_1, \beta_1; z))',\\
&D^{p,m}_\lambda(\alpha_1, \beta_1)f(z) = D^{p,1}_\lambda(D^{p,m-1}_\lambda(\alpha_1, \beta_1)f(z)),
\end{aligned} \tag{5}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geq 0$.
If $f(z) \in \mathscr{A}_p$, then we have

$$D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z) = z^p + \sum_{n=1}^{\infty}\left(\frac{p+\lambda n}{p}\right)^m \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_s)_n} a_n \frac{z^{p+n}}{n!}. \tag{6}$$

It can be seen that, by specializing the parameters the operator $D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)$ reduces to many known and new integral and differential operators. In particular, when $m=0$ and $p=1$ the operator $D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)$ reduces to the well known Dziok- Srivastava operator [3] and for $p=1$, $q=2$, $s=1$, $\alpha_1=\beta_1$, and $\alpha_2=1$, it reduces to the operator introduced by F. AL-Oboudi [1]. Further we remark that, when $p=1$, $q=2$, $s=1$, $\alpha_1=\beta_1$, $\alpha_2=1$, and $\lambda=1$ the operator $D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)$ reduces to the operator introduced by G. S. Sălăgean [8].

It can be easily verified from (6) that

$$\lambda z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))' = pD_{\lambda}^{p,m+1}(\alpha_1,\beta_1)f(z) - p(1-\lambda)D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z). \tag{7}$$

Using the operator $D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)$ we now define the following class of p-valent analytic functions.

Definition 1. *A function $f(z) \in \mathscr{A}_p$ is said to be in the class $S_{\lambda,m}^{p,j}(A,B;\gamma)$ of p-valent functions of complex order $\gamma \neq 0$ in $\mathscr{U}$ if and only if*

$$Re\left\{1+\frac{1}{\gamma}\left(\frac{z\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)\right)^{(j+1)}}{\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)\right)^{(j)}} - p + j\right)\right\} \prec \frac{1+Az}{1+Bz} \tag{8}$$

$$(z \in \mathscr{U};\, -1 \leq B < A \leq 1;\, p \in \mathbb{N};\, m,j \in \mathbb{N}_0;\, \gamma \in \mathbb{C}-\{0\};\, |\gamma\lambda(A-B)+pB| \leq p).$$

It can be seen that, by specializing the parameters the class $S_{\lambda,m}^{p,j}(A,B;\gamma)$ reduces to many known subclasses of analytic functions. In particular, when $A=1$ and $B=-1$ the class reduces to the class $S_{\lambda,m}^{p,j}(\gamma)$ which has recently been introduced by C.Selvaraj and K.A.Selvakumaran [9]. Further, when $q=2$, $s=1$, $\alpha_1=\beta_1$, and $\alpha_2=1$, we have the following relationships:

(1) $S_{\lambda,0}^{1,0}(1,-1;\gamma) = \mathscr{S}(\gamma) \quad (\gamma \in \mathbb{C}-\{0\})$.

(2) $S_{\lambda,0}^{1,1}(1,-1;\gamma) = \mathscr{K}(\gamma) \quad (\gamma \in \mathbb{C}-\{0\})$.

(3) $S_{\lambda,0}^{1,0}(1,-1;1-\alpha) = \mathscr{S}^*(\alpha)$ for $0 \leq \alpha < 1$.

The classes $\mathscr{S}(\gamma)$ and $\mathscr{K}(\gamma)$ are said to be the classes of starlike and convex functions of complex order $\gamma \neq 0$ in $\mathscr{U}$ which were studied by M. A. Nasr and M. K. Aouf [6] and P. Wiatrowski [10] and $\mathscr{S}^*(\alpha)$ is the class of starlike functions of order α in $\mathscr{U}$.

2. Majorization Problem for the Class $S_{\lambda,m}^{p,j}(A,B;\gamma)$

Theorem 1. *Let the function $f(z)$ be in the class $\mathscr{A}_p$ and suppose that $g(z) \in S_{\lambda,m}^{p,j}(A,B;\gamma)$. If $\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)\right)^{(j)}$ is majorized by $\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)}$ in $\mathscr{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\left(D_{\lambda}^{p,m+1}(\alpha_1,\beta_1)f(z)\right)^{(j)}\right| \leq \left|\left(D_{\lambda}^{p,m+1}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \text{ for } |z| \leq r_1, \tag{9}$$

where $r_1 = r_1(p,\gamma,\lambda,A,B)$ is the smallest positive root of the equation

$$|\gamma\lambda(A-B)+pB|r^3 - (p+2\lambda|B|)r^2 - (|\gamma\lambda(A-B)+pB|+2\lambda)r + p = 0 \tag{10}$$

$$(-1 \leq B < A \leq 1; p \in \mathbb{N};\ \gamma \in \mathbb{C}-\{0\};\ \lambda \geq 0).$$

Proof. Let

$$h(z) = 1 + \frac{1}{\gamma}\left(\frac{z\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j+1)}}{\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)}} - p + j\right) \tag{11}$$

$$(p \in \mathbb{N};\ m,j \in \mathbb{N}_0;\ \gamma \in \mathbb{C}-\{0\};\ p > j).$$

Since $g(z) \in S_{\lambda,m}^{p,j}(\gamma)$, we find from (8) that

$$h(z) = \frac{1+Aw(z)}{1+Bw(z)}, \tag{12}$$

where $w(z)$ is analytic in $\mathscr{U}$, which satisfies the conditions

$$w(0) = 0 \text{ and } |w(z)| < 1 \qquad (z \in \mathscr{U}).$$

It follows from (11) and (12)that

$$\frac{z\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j+1)}}{\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)}} = \frac{(p-j)+[\gamma(A-B)+(p-j)B]w(z)}{1+Bw(z)} \tag{13}$$

In view of

$$\lambda z(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))^{(j+1)} = p(D_{\lambda}^{p,m+1}(\alpha_1,\beta_1)f(z))^{(j)} - (p-p\lambda+\lambda j)(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z))^{(j)}, \tag{14}$$

(13) immediately yields the following inequality:

$$\left|\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \leq \frac{p(1+|B||z|)}{p-|\gamma\lambda(A-B)+pB||z|}\left|\left(D_{\lambda}^{p,m+1}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right|. \tag{15}$$

Since $\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)\right)^{(j)}$ is majorized by $\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)}$ in $\mathscr{U}$, there exist an analytic function $\varphi(z)$ such that

$$\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)f(z)\right)^{(j)} = \varphi(z)\left(D_{\lambda}^{p,m}(\alpha_1,\beta_1)g(z)\right)^{(j)} \tag{16}$$

and $|\varphi(z)| \le 1 \quad (z \in \mathscr{U})$. Thus we have

$$z\big(D_\lambda^{p,m}(\alpha_1,\beta_1)f(z)\big)^{(j+1)} = z\varphi'(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)} + z\varphi(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j+1)}. \tag{17}$$

Using (14), in the above equation, we get

$$\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z)\big)^{(j)} = \frac{\lambda z}{p}\varphi'(z)\big(D_\lambda^{p,m}(\alpha_1,\beta_1)g(z)\big)^{(j)} + \varphi(z)\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\big)^{(j)}. \tag{18}$$

Noting that $\varphi(z)$ satisfies (cf. [4, 7])

$$|\varphi'(z)| \le \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in \mathscr{U}), \tag{19}$$

we see that

$$\begin{aligned} &\left|\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z)\big)^{(j)}\right| \\ &\le \left\{\varphi(z) + \frac{1-|\varphi(z)|^2}{1-|z|^2}\,\frac{\lambda|z|(1+|B||z|)}{p-|\gamma\lambda(A-B)+pB||z|}\right\}\left|\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\big)^{(j)}\right| \end{aligned} \tag{20}$$

which, upon setting

$$|z| = r, \quad \text{and} \quad |\varphi(z)| = \rho \qquad (0 \le \rho \le 1)$$

leads us to the following inequality:

$$\left|\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)f(z)\big)^{(j)}\right| \le \frac{\Theta(\rho)}{(1-r^2)(p-|\gamma\lambda(A-B)+pB|r)}\left|\big(D_\lambda^{p,m+1}(\alpha_1,\beta_1)g(z)\big)^{(j)}\right|, \tag{21}$$

where the function $\Theta(\rho)$ defined by

$$\Theta(\rho) := -\lambda r(1+|B|r)\rho^2 + (1-r^2)(p-|\gamma\lambda(A-B)+pB|r)\rho + \lambda r(1+|B|r) \quad (0 \le \rho \le 1)$$

takes its maximum value at $\rho = 1$ with $r = r_1(p,\gamma,\lambda,A,B)$, the smallest positive root of the equation (10). Furthermore, if $0 \le \sigma \le r_1(p,\gamma,\lambda,A,B)$, then the function

$$\Phi(\rho) := -\lambda\sigma(1+|B|\sigma)\rho^2 + (1-\sigma^2)(p-|\gamma\lambda(A-B)+pB|\sigma)\rho + \lambda\sigma(1+|B|\sigma)$$

increases in the interval $0 \le \rho \le 1$, so that $\Phi(\rho)$ does not exceed

$$\Phi(1) = (1-\sigma^2)(p-|\gamma\lambda(A-B)+pB|\sigma) \qquad (0 \le \sigma \le r_1(p,\gamma,\lambda,A,B)).$$

Therefore, from this fact, (21) gives the inequality (9).

As a special case of Theorem 1, when $A = 1$ and $B = -1$, we have

Corollary 1. *[9] Let the function $f(z)$ be in the class $\mathscr{A}_p$ and suppose that $g(z) \in S^{p,j}_{\lambda,m}(\gamma)$. If $\left(D^{p,m}_{\lambda}(\alpha_1,\beta_1)f(z)\right)^{(j)}$ is majorized by $\left(D^{p,m}_{\lambda}(\alpha_1,\beta_1)g(z)\right)^{(j)}$ in $\mathscr{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|\left(D^{p,m+1}_{\lambda}(\alpha_1,\beta_1)f(z)\right)^{(j)}\right| \leq \left|\left(D^{p,m+1}_{\lambda}(\alpha_1,\beta_1)g(z)\right)^{(j)}\right| \text{ for } |z| \leq r_1, \tag{22}$$

where

$$r_1 = r_1(p,\gamma,\lambda) := \frac{k-\sqrt{k^2-4p|2\gamma\lambda-p|}}{2|2\gamma\lambda-p|} \tag{23}$$
$$(k := 2\lambda+p+|2\gamma\lambda-p|; p \in \mathbb{N};\ \gamma \in \mathbb{C}-\{0\};\ \lambda \geq 0).$$

Setting $A=1$, $B=-1$, $p=1$ and $j=0$ in Theorem 1, we have

Corollary 2. *Let the function $f(z) \in \mathscr{A}$ be analytic and univalent in the open unit disk $\mathscr{U}$ and suppose that $g(z) \in S^{1,0}_{\lambda,m}(\gamma)$. If $\left(D^{1,m}_{\lambda}(\alpha_1,\beta_1)f(z)\right)$ is majorized by $\left(D^{1,m}_{\lambda}(\alpha_1,\beta_1)g(z)\right)$ in $\mathscr{U}$, then*

$$\left|\left(D^{1,m+1}_{\lambda}(\alpha_1,\beta_1)f(z)\right)\right| \leq \left|\left(D^{1,m+1}_{\lambda}(\alpha_1,\beta_1)g(z)\right)\right| \quad \text{for} \quad |z| \leq r_2, \tag{24}$$

where

$$r_2 := \frac{k-\sqrt{k^2-4|2\gamma\lambda-1|}}{2|2\gamma\lambda-1|} \tag{25}$$
$$(k := 2\lambda+1+|2\gamma\lambda-1|;\ \gamma \in \mathbb{C}-\{0\};\ \lambda \geq 0).$$

Further putting $\lambda=1$, $m=0$, $q=2$, $s=1$, $\alpha_1=\beta_1$, and $\alpha_2=1$ in Corollary 2, we get

Corollary 3. *[2] Let the function $f(z) \in \mathscr{A}$ be analytic and univalent in the open unit disk $\mathscr{U}$ and suppose that $g(z) \in \mathscr{S}(\gamma)$. If $f(z)$ is majorized by $g(z)$ in $\mathscr{U}$, then*

$$\left|f'(z)\right| \leq \left|g'(z)\right| \quad \text{for} \quad |z| \leq r_3, \tag{26}$$

where

$$r_3 := \frac{3+|2\gamma-1|-\sqrt{9+2|2\gamma-1|+|2\gamma-1|^2}}{2|2\gamma-1|}. \tag{27}$$

For $\gamma=1$, Corollary 3 reduces to the following result:

Corollary 4. *[5] Let the function $f(z) \in \mathscr{A}$ be analytic and univalent in the open unit disk $\mathscr{U}$ and suppose that $g(z) \in \mathscr{S}^* = \mathscr{S}^*(0)$. If $f(z)$ is majorized by $g(z)$ in $\mathscr{U}$, then*

$$\left|f'(z)\right| \leq \left|g'(z)\right| \quad \text{for} \quad |z| \leq 2-\sqrt{3}. \tag{28}$$

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