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Certain Sufficient Conditions for Univalence of Two Integral Operators

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Abstract. In this paper, we obtain new sufficient conditions for two general integral operators to be univalent in the open unit disc. A number of new univalent conditions would follow upon specializing the parameters involved in our main results.

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1. Introduction and Definitions

Let $\mathscr{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathscr{U} = \{z : |z| < 1\}$. Further, by $\mathscr{S}$ we shall denote the class of all functions in $\mathscr{A}$ which are univalent in $\mathscr{U}$.

In [10] Ozaki and Nunokawa showed that if $f \in \mathscr{A}$ and

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq |z|^2, \text{ for all } z \in \mathscr{U}, \tag{1}$$

then the function f is univalent in $\mathscr{U}$.

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Making use of the univalence criteria (1), several authors (e.g., see [1, 3, 4, 6, 8, 14, 16, 17]), obtained many sufficient conditions for the univalency of the integral operators

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)=\left\{\int_0^z \beta t^{\beta-1}\left(\frac{f_1(t)}{t}\right)^{\frac{1}{\alpha_1}}\cdots\left(\frac{f_n(t)}{t}\right)^{\frac{1}{\alpha_n}}dt\right\}^{\frac{1}{\beta}} \tag{2}$$

and

$$G_{n,\beta}(z)=\left\{[n(\beta-1)+1]\int_0^z \left(f_1(t)\right)^{\beta-1}\cdots\left(f_n(t)\right)^{\beta-1}dt\right\}^{\frac{1}{n(\beta-1)+1}} \tag{3}$$

where the functions $f_1, f_2, \ldots, f_n$ belong to the class $\mathscr{A}$ and the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β are complex numbers such that the integrals in (2) and (3) exist. Here and throughout in the sequel every many-valued function is taken with the principal branch. The integral operator in (2) was introduced and studied by Seenivasagan and Breaz [15], and the integral operator in (3) was introduced and studied by Breaz and Breaz [2].

In this paper we are mainly interested on some integral operators of the type (2) and (3). More precisely, we obtain new sufficient conditions for this operators to be univalent in the open unit disc $\mathscr{U}$.

In the proofs of our main results we need the following univalence criteria. The first result, i.e. Lemma 1 is a generalization of Ozaki - Nunokawa's criterion (1) obtained by Raducanu *et al.* [13], while the second, i.e. Lemma 2 is a generalization of Ahlfors' and Becker's univalence criterion [11]. Finally, we need the well-known general Schwarz Lemma.

Lemma 1 ([13]). *Let $f \in \mathscr{A}$ and $m > 0$ such that*

$$\left|\left(\frac{z^2 f'(z)}{f^2(z)}-1\right)-\frac{m-1}{2}|z|^{m+1}\right|\le\frac{m+1}{2}|z|^{m+1}, \tag{4}$$

for all $z \in \mathscr{U}$. Then the function f is analytic and univalent in $\mathscr{U}$.

Lemma 2 ([11]). *Let $\beta \in \mathbb{C}$ with $Re(\beta) > 0$, $c \in \mathbb{C}$ with $|c| \le 1$, $c \neq -1$. If $h \in \mathscr{A}$ satisfies*

$$\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right|\le 1,$$

for all $z \in \mathscr{U}$, then the integral operator

$$F_\beta(z)=\left\{\beta\int_0^z t^{\beta-1}h'(t)dt\right\}^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathscr{U}$.

Lemma 3 ([9]). *Let the function f be regular in the disk $\mathscr{U}_R = \{z : |z| < R\}$, with $\left|f(z)\right| < M$ for fixed M. If $f(z)$ has one zero with multiplicity order bigger than m for $z = 0$, then*

$$\left|f(z)\right| \leq \frac{M}{R^m}|z|^m \qquad (z \in \mathscr{U}_R).$$

The equality can hold only if

$$f(z) = e^{i\theta}(M/R^m)z^m$$

where θ is constant.

2. Univalence Conditions for $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$

We first prove

Theorem 1. *Let $\alpha_i \in \mathbb{C}$, $M_i \geq 1$, $m_i > 0$ for all $i = 1,\ldots,n$ and $\beta \in \mathbb{C}$ with*

$$Re(\beta) \geq \sum_{i=1}^{n} \frac{(m_i+1)M_i+1}{\left|\alpha_i\right|}. \tag{5}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 - \frac{1}{Re(\beta)}\sum_{i=1}^{n} \frac{(m_i+1)M_i+1}{\left|\alpha_i\right|}. \tag{6}$$

If $f_i \in \mathscr{A}$ ($i = 1,\ldots,n$) satisfies the inequality (4) and

$$\left|f_i(z)\right| \leq M_i \ (z \in \mathscr{U}, i = 1,\ldots,n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z)$ defined by (2) is analytic and univalent in $\mathscr{U}$.

Proof. Define

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt$$

we observe that $h(0) = h'(0) - 1 = 0$. On the other hand, it is easy to see that

$$h'(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}. \tag{7}$$

Now we differentiate (7) logarithmically and multiply by z on both sides, we obtain

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{zf_i{}'(z)}{f_i(z)} - 1\right). \tag{8}$$

Since $\left|f_i(z)\right| \leq M_i$ $(z \in \mathscr{U},\ i = 1,\ldots,n)$, then by the general Schwarz Lemma, we obtain $\left|f_i(z)\right| \leq M_i\,|z|$ for all $z \in \mathscr{U}$ and $i = 1,\ldots,n$, we thus from (4) and (8) find that

$$\begin{aligned}
\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{\left|\alpha_i\right|}\left(\left|\frac{zf_i{}'(z)}{f_i(z)}\right|+1\right)\\
&= \sum_{i=1}^{n}\frac{1}{\left|\alpha_i\right|}\left(\left|\frac{z^2f_i{}'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right|+1\right)\\
&\leq \sum_{i=1}^{n}\frac{1}{\left|\alpha_i\right|}\left(\left|\left(\frac{z^2f_i{}'(z)}{[f_i(z)]^2}-1\right)-\frac{m_i-1}{2}\,|z|^{m_i+1}\right|M_i+\left(1+\frac{m_i-1}{2}\,|z|^{m_i+1}\right)M_i+1\right)\\
&\leq \sum_{i=1}^{n}\frac{1}{\left|\alpha_i\right|}\left(\frac{m_i+1}{2}\,|z|^{m_i+1}M_i+\left(1+\frac{m_i-1}{2}\,|z|^{m_i+1}\right)M_i+1\right)\\
&\leq \sum_{i=1}^{n}\frac{(m_i+1)M_i+1}{\left|\alpha_i\right|}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| &\leq |c|+\frac{1}{\left|\beta\right|}\left|\frac{zh''(z)}{h'(z)}\right|\\
&\leq |c|+\frac{1}{\left|\beta\right|}\sum_{i=1}^{n}\frac{(m_i+1)M_i+1}{\left|\alpha_i\right|}\\
&\leq |c|+\frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}\frac{(m_i+1)M_i+1}{\left|\alpha_i\right|},
\end{aligned}$$

which, in the light of the hypothesis (6), yields

$$\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| \leq 1.$$

Finally, by applying Lemma 2, we conclude that $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) \in \mathscr{S}$.

Letting $m_1 = m_2 = \cdots = m_n = m$ in Theorem 1, we have

Corollary 1. *Let $\alpha_i \in \mathbb{C}$ $M_i \geq 1$ for all $i = 1,\ldots,n$, $m > 0$ and $\beta \in \mathbb{C}$with*

$$Re(\beta) \geq \sum_{i=1}^{n}\frac{(m+1)M_i+1}{\left|\alpha_i\right|}. \tag{9}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1-\sum_{i=1}^{n}\frac{(m+1)M_i+1}{\left|\alpha_i\right|}. \tag{10}$$

If $f_i \in \mathscr{A}$ $(i=1,\dots,n)$ satisfies the inequality (4) and

$$\left|f_i(z)\right| \le M_i \ (z \in \mathscr{U}, i=1,\dots,n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\dots,\alpha_n,\beta}(z)$ defined by (2) is analytic and univalent in $\mathscr{U}$.

Remark 1. *If we put $m=1$ in Corollary 1, we obtain Theorem 2.1 in [5].*

Letting $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$ and $M_1=M_2=\cdots=M_n=M$ in Corollary 1, we have

Corollary 2. *Let $\alpha \in \mathbb{C}$, $M \ge 1$, $m>0$ and $\beta \in \mathbb{C}$ with*

$$Re(\beta) \ge \frac{n(m+1)M+n}{|\alpha|}. \tag{11}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \le 1 - \frac{n(m+1)M+n}{|\alpha| Re(\beta)}. \tag{12}$$

If $f_i \in \mathscr{A}$ $(i=1,\dots,n)$ satisfies the inequality (4) and

$$\left|f_i(z)\right| \le M \ (z \in \mathscr{U}, i=1,\dots,n),$$

then the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \int_0^z \beta t^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathscr{U}$.

Letting $n=1$, $\alpha_1=\alpha$, $M_1=M$, $m_1=m$ and $f_1=f$ in Theorem 1, we have

Corollary 3. *Let $\alpha \in \mathbb{C}$, $M \ge 1$, $m>0$ and $\beta \in \mathbb{C}$ with*

$$Re(\beta) \ge \frac{(m+1)M+1}{|\alpha|}. \tag{13}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \le 1 - \frac{(m+1)M+1}{|\alpha| Re(\beta)}. \tag{14}$$

If $f \in \mathscr{A}$ satisfies the inequality (4) and

$$\left|f(z)\right| \le M \ (z \in \mathscr{U}),$$

then the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \int_0^z \beta t^{\beta-1} \left(\frac{f(t)}{t}\right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}}$$

defined by (2) is analytic and univalent in $\mathscr{U}$.

3. Univalence Conditions for $G_{n,\beta}(z)$

Next, we prove

Theorem 2. *Let $M_i \geq 1$, $m_i > 0$ for all $i = 1, \dots, n$ and $\beta \geq 1$ with*

$$\left(\frac{\beta-1}{\beta}\right)\sum_{i=1}^{n}[(m_i+1)M_i+1] \leq 1. \tag{15}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq 1 + \left(\frac{1-\beta}{\beta}\right)\sum_{i=1}^{n}[(m_i+1)M_i+1]. \tag{16}$$

If $f_i \in \mathscr{A}$ $(i=1,\dots,n)$ satisfies the inequality (4) and

$$\left|f_i(z)\right| \leq M_i \ (z \in \mathscr{U}, i = 1, \dots, n),$$

then the integral operator $G_{n,\beta}(z)$ defined by (2) is analytic and univalent in $\mathscr{U}$.

Proof. Setting

$$h(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{f_i(t)}{t}\right)^{\beta-1} dt$$

so that,

$$h'(z) = \prod_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\beta-1}. \tag{17}$$

and

$$h''(z) = (\beta-1).\sum_{i=1}^{n}\left(\frac{f_i(z)}{z}\right)^{\beta-2}\left(\frac{zf_i - f_i}{z^2}\right).\prod_{k=1(k\neq i)}^{n}\left(\frac{f_k(z)}{z}\right)^{\beta-1}. \tag{18}$$

It follows from (17) and (18) that

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n}(\beta-1)\left(\frac{zf_i{}'(z)}{f_i(z)} - 1\right). \tag{19}$$

Since $\left|f_i(z)\right| \leq M_i \, (z \in \mathscr{U},\ i = 1, \dots, n)$, then by the general Schwarz Lemma, we obtain $\left|f_i(z)\right| \leq M_i\,|z|$ for all $z \in \mathscr{U}$ and $i = 1, \dots, n$, we thus from (4) and (19) find that

$$\begin{aligned}
\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}(\beta-1)\left(\left|\frac{zf_i{}'(z)}{f_i(z)}\right| + 1\right) \\
&\leq \sum_{i=1}^{n}(\beta-1)\left(\left|\frac{z^2f_i{}'(z)}{[f_i(z)]^2}\right|\left|\frac{f_i(z)}{z}\right| + 1\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n}(\beta-1)\left(\left|\left(\frac{z^2 f_i'(z)}{[f_i(z)]^2}-1\right)-\frac{m_i-1}{2}|z|^{m_i+1}\right| M_i+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)M_i+1\right)\\
&\leq \sum_{i=1}^{n}(\beta-1)\left(\frac{m_i+1}{2}|z|^{m_i+1}M_i+\left(1+\frac{m_i-1}{2}|z|^{m_i+1}\right)M_i+1\right)\\
&\leq \sum_{i=1}^{n}(\beta-1)[(m_i+1)M_i+1].
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right| &\leq |c|+\frac{1}{\beta}\left|\frac{zh''(z)}{h'(z)}\right|\\
&\leq |c|+\left(\frac{\beta-1}{\beta}\right)\sum_{i=1}^{n}[(m_i+1)M_i+1],
\end{aligned}
$$

which, in the light of the hypothesis (16), yields

$$
\left|c\,|z|^{2\beta}+(1-|z|^{2\beta})\frac{zh''(z)}{\beta h'(z)}\right|\leq 1.
$$

Finally, by applying Lemma 2, we conclude that $G_{n,\beta}(z)\in\mathcal{S}$.

Letting $m_1=m_2=\ldots=m_n=m$ and $M_1=M_2=\ldots=M_n=M$ in Theorem 2, we have

Corollary 4. *Let $M\geq 1$, $m>0$ and $\beta\in\mathbb{R}$ with*

$$
\beta\in\left[1,\frac{((m+1)M+1)n}{((m+1)M+1)n-1}\right]. \tag{20}
$$

and let $c\in\mathbb{C}$ be such that

$$
|c|\leq 1+\left(\frac{1-\beta}{\beta}\right)((m+1)M+1)n. \tag{21}
$$

If $f_i\in\mathcal{A}$ $(i=1,\ldots,n)$ satisfies the inequality (4) and

$$
\left|f_i(z)\right|\leq M\ \ (z\in\mathcal{U})
$$

then the integral operator $G_{n,\beta}(z)$ defined by (2) is analytic and univalent in $\mathcal{U}$.

Remark 2. *If we put $m=1$ in Corollary 4, we obtain Theorem 4 in [7].*

If we put $m=M=n=1$ and $f_1=f$ in Corollary 4, we obtain the following interesting result obtained by Pescar [12].

Corollary 5 ([12]). *Let $\beta \in \mathbb{R}$ with*

$$\beta \in \left[1, \frac{3}{2}\right]. \tag{22}$$

and let $c \in \mathbb{C}$ be such that

$$|c| \leq \frac{3-2\beta}{\beta} \ (c \neq -1). \tag{23}$$

If $f \in \mathscr{A}$ satisfies the inequality (4) and

$$\left|f(z)\right| \leq 1 \ (z \in \mathscr{U})$$

then the integral operator $G_\beta(z)$ defined by

$$G_\beta(z) = \left(\beta \int_0^z \left(f(t)\right)^{\beta-1} dt\right)^{\frac{1}{\beta}}$$

is analytic and univalent in $\mathscr{U}$.

References

[1] D. Breaz, Univalence properties for a general integral operator, *Bull. Korean Math.Soc.* 46 (3), 439-446. 2009.

[2] D. Breaz and N. Breaz, Univalence of an integral operator, *Mathematica (Cluj)* 47 (70), 35-38. 2005.

[3] D. Breaz and N. Breaz, Sufficient univalence conditions for analytic functions, *J. Ineq. Appl.*, Volume 7, Article ID 86493, 5 pages. 2007.

[4] D. Breaz and N. Breaz, An integral univalent operator, *Acta Math. Univ. Comenianac,* Vol. LXXVI, 2, 137-142. 2007.

[5] N. Breaz and D. Breaz, Sufficient univalent conditions for an integral operator, P*roc. Int. Symp. on new development of geometric functions theory and its applications GFTA*, 60-63. 2008.

[6] D. Breaz and S. Owa, Some extensions of univalent conditions for certain integral operators, *Math. Ineq. Appl.* Vol. 10, 2, 321-325. 2007.

[7] D. Breaz, N. Breaz and H. M. Srivastava, An extension of the univalent condition for a family of integral operators, *Appl. Math. Lett*. 22 (1), 41-44. 2009.

[8] S. Bulut, Univalence condition for a new generalization of the family of integral operators, *Acta Universitatis Apulensis*, No. 18, 71-78. 2009.

[9] Z. Nehari, Conformal mapping, McGraw-Hill Book Comp., New York, 1952.

[10] S. Ozaki and M. Nunokawa, The Schwarzian derivative and univalent functions, *Proc. Amer. Math. Soc.* 33, 392–394. 1972.

[11] V. Pescar, A new generalization of Ahlfor's and Becker's criterion of univalence, *Bull. Malaysian Math. Soc. (Second Series)* 19, 53-54. 1996.

[12] V. Pescar, On the univalence of two integral operators, *J. Indian Acad. Math.* 27, 239-243. 2005.

[13] D. Raducanu, I. Radomir, M. E. Gageonea, N. R. Pascu, A Generalization of Ozaki–Nunokawa's Univalence Criterion, *J. Inequal. Pure Appl. Math,* Vol.

[14] N. Seenivasagan, Sufficient conditions for univalence, *Applied Math.E-Notes*, 8, 30-35. 2008.

[15] N. Seenivasagan and D. Breaz, Certain sufficient conditions for univalence, *General Math.*15 (4), 7-15. 2007.

[16] C. Selvaraj and K. R. Karthikeyan, Sufficient conditions for univalence of a general integral operator, *Acta Universitatis Apulensis*, No. 17, 87-94. 5, Issue 4, Paper No. 95 (2004).

[17] H.M. Srivastava, E. Deniz and H. Orhan, Some general univalence criteria for a family of integral operators, *Appl. Math. Comp*., 215, 3696-3701. 2010.