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Sharp Bounds for the Probability of the Union of Events Under Unimodality Condition

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Abstract. Linear programming problem is formulated for bounding the probability of the union of events, where the probability distribution of the occurrences is supposed to be unimodal with known mode and some of the binomial moments of the events are also known. Using a theorem on combinatorial determinants the dual feasible bases of a relaxed problem are fully described. The bounds for the probability of the union are presented in the form of formulas as well as the results of customized algorithmic solution of the LP's involved.

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Key words: Binomial moment problem, linear programming, bounding probabilities, discrete unimodality, reliability.

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1. Introduction

Let $A_1, ..., A_n$ be arbitrary events in an arbitrary probability space. The kth binomial moment of them is designated by S_k and is defined by the equation:

$$S_k = \sum_{1 \le i_1 < ... < i_k \le n} P(A_{i_1} ... A_{i_k}) \ , \quad k = 1, ..., n \ .$$

Let $S_0 = 1$. It is well known that (see, e.g., Prékopa [11])

$$S_k = E\left[\binom{\nu}{k}\right] , \quad k = 0, ..., n \ , \tag{1.1}$$

where ν is the number of those events that occur.

If we introduce the notation $p_k = P(\nu = k), \ k = 0, ..., n$, then we can write (1.1) in the following more detailed form:

$$S_k = \sum_{i=0}^{n} \binom{i}{k} p_i \ , \quad k = 0, ..., n \ .$$

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To compute the probability of the union of the events the inclusion-exclusion formula is available:

$$P(A_1 \cup ... \cup A_n) = S_1 - S_2 + ... + (-1)^{n-1} S_n \ .$$

However, if n is large, we may not be able to compute all the binomial moments, still, we may be able to compute a few of them. Given that, and further information about the probability of the random variable ν, we can give lower and upper bounds for the probability of the union. The bounds may serve for approximation of that probability provided that they are close to each other.

In this paper we assume that $S_{k_1}, ..., S_{k_m}$ are known for some $1 \le k_1 < ... < k_m$, $m < n$. We do not assume the knowledge of the probability distribution $\{p_i\}$ but we assume that it is unimodal, i.e., there exists an integer M $(0 \le M \le n)$ such that $p_0 \le ... \le p_M$, $p_M \ge ... \ge p_n$. The number M may be equal to 0 or n, or satisfy $0 < M < n$.

To obtain lower and upper bounds for the probability of the union of events we formulate the LP:

$$\min(\max) \sum_{i=1}^{n} p_i$$

subject to

$$\sum_{i=0}^{n} \binom{i}{k_j} p_i = S_{k_j} \ , \quad j = 0, ..., m \tag{1.2}$$

$$p_0 \le ... \le p_M$$

$$p_M \ge ... \ge p_n$$

$$p_i \ge 0 \ , \quad i = 0, ..., n \ ,$$

where $k_0 = 0$. In problem (1.2) the $p_0, ..., p_n$ are unknown variables. If $m < n$, then there are infinitely many probability distributions satisfying the constraints of problem (1.2). One of them is the true distribution of ν. This implies that the optimum value of the $\min$ $(\max)$ problem (1.2) is a lower (upper) bound for the probability of the union. These bounds have the property that, given $S_{k_1}, ..., S_{k_m}$ and the knowledge of the unimodality of $\{p_i\}$, no better bounds can be given for $P(A_1 \cup ... \cup A_n)$. In view of this fact, we call them sharp bounds. The binomial moment problem, without the unimodality constraint, has extensively been studied. Prékopa [7–10] has shown that the sharp Bonferroni inequalities of Dawson and Sankoff [4] and others can be formulated as linear programming problems. For the case of $m \le 3$ and $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, Kwerel [6] has already used linear programming to obtain sharp bounds for the probability of the union. He fully described the dual feasible bases of the problem, also in the cases, where we are bounding the probabilities that at least r and exactly r events occur, reproduced known formulas this way and gave special dual type algorithms to solve the problems. Boros and Prékopa [1] exploited the linear programming methodology and derived a variety of sharp bounds of Boolean functions of events. The list of other papers presenting bounds along this line includes Prékopa, Gao [13], Bukszár [3], Bukszár, Prékopa [2].

The paper by E. Subasi, M. Subasi and A. Prékopa [14] is the first, where sharp bounds are presented for the probability of the union under unimodality constraint for the distribution of the random variable ν. In that paper problem (1.2) was used for the case of $m = 2$, $k_1 = 1$, $k_2 = 2$ and bounds are given by the use of formulas as well as by the dual algorithm of linear programming. In another paper by E. Subasi, M. Subasi and A. Prékopa, bounding formulas have been obtained for the probability that at least r and exactly r out of n events occur, under the same conditions.

The purpose of the present paper is to derive a general theorem in connection with problem (1.2) that characterizes the dual feasible bases of a relaxed version of the problem, further, to present closed form and algorithmic bounds for the probability of the union.

As it is known in linear programming theory, the objective function value corresponding to any dual feasible basis in the minimization (maximization) problem provides us with a lower (upper) bound for the optimum value of the problem.

First we reformulate problem (1.2) by introducing new variables $v_0, ..., v_n$. This can be done in two different ways:

$$p_0 = v_0\ ,\ \ p_1 = v_0 + v_1\ ,\ \dots\ ,\ \ p_M = v_0 + ... + v_M$$

$$p_{M+1} = v_{M+1} + ... + v_n\ ,\ p_{M+2} = v_{M+2} + ... + v_n\ ,\ \dots\ ,\ \ p_n = v_n\ , \tag{1.3}$$

and

$$p_0 = v_0\ ,\ \ p_1 = v_0 + v_1\ ,\ \dots\ ,\ \ p_{M-1} = v_0 + ... + v_{M-1}$$

$$p_M = v_M + ... + v_n\ ,\ p_{M+1} = v_{M+1} + ... + v_n\ ,\ \dots\ ,\ \ p_n = v_n\ . \tag{1.4}$$

The case $M = n$ is included in (1.3) and the case $M = 0$ included in (1.4).
If we use representation (1.3) in problem (1.2), we obtain the following problem:

$$min(max)\ \{Mv_0 + \sum_{i=1}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n}(i-M)v_i = 1 \tag{1.5}$$

$$\sum_{i=0}^{M}\left[\binom{i}{k_j} + ... + \binom{M}{k_j}\right]v_i + \sum_{i=M+1}^{n}\left[\binom{M+1}{k_j} + ... + \binom{i}{k_j}\right]v_i = S_{k_j}\ ,$$

$$j = 1, ..., m$$

$$v_0 + ... + v_M - v_{M+1} - ... - v_n \geq 0 \qquad (1.5a)$$

$$v_i \geq 0\ ,\ \ i = 0, ..., n\ .$$

In case of representation (1.4) the problem can be formulated as follows:

$$min(max)\ \{(M-1)v_0 + \sum_{i=1}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i\}$$

subject to

$$\sum_{i=0}^{M-1}(M-i)v_i + \sum_{i=M}^{n}(i-M+1)v_i = 1 \qquad (1.6)$$

$$\sum_{i=0}^{M-1}\left[\binom{i}{k_j}+\dots+\binom{M-1}{k_j}\right]v_i + \sum_{i=M}^{n}\left[\binom{M}{k_j}+\dots+\binom{i}{k_j}\right]v_i = S_{k_j}\ , \quad j=1,\dots,m$$

$$v_M+\dots+v_n - v_0 - \dots - v_{M-1} \ge 0 \qquad (1.6a)$$

$$v_i \ge 0\ ,\quad i=0,\dots,n\ .$$

Problem (1.5) without the constraint (1.5a) and problem (1.6) without (1.6a) will be called relaxed problems. For both relaxed problems $A=(a_0,\dots,a_n)$ will designate the matrix of the equality constraints, b the right hand side vector and c the vector of coefficients of the objective function.

The organization of the paper is as follows. In Section 2 we characterize the dual feasible bases of the relaxed problem. In Section 3 bounding formulas are derived for the probability of the union, for the case of $m=2$ and general k_1, k_2 $(1 \le k_1 < k_2 \le n)$. In Section 4 we present closed form bounds for the case of $m=3$ and $k_1=1$, $k_2=2$, $k_3=3$. In Section 5 upper bound formulas are derived for the probability of the union, for the case of $m=4$ and $k_1=1$, $k_2=2$, $k_3=3$, $k_4=4$. In Section 6 general algorithms are presented to obtain algorithmic bounds. In Section 7 we present an application of our bounding methodology, where shape information about the unknown probability distribution can be used. Finally, numerical examples are presented in Section 8.

2. Characterization of the dual feasible bases of the relaxed problem

In what follows we make use of a general theorem for the Pascal matrix, i.e., the matrix P consisting of binomial coefficients:

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ & 1 & 2 & 3 & \cdots & n-1 & n \\ & & 1 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ & & & & \ddots & \vdots & \vdots \\ & & & & & 1 & \binom{n}{n-1} \\ & & & & & & 1 \end{pmatrix},$$

where there are zeros in the unfilled positions.

The term "minor" of a matrix will be used in the following sense: it is the determinant of a submatrix crossed out arbitrarily by the same number of rows and columns.

Theorem 2.1. [5, 7] *Any minor of P that has all positive entries in its main diagonal, is positive.*

In the next theorem we characterize the dual feasible bases of the relaxed version of problems (1.5), (1.6). For basic notions, facts and algorithms in connection with linear programming the reader is referred to the paper by Prékopa [12].

Theorem 2.2. *Any dual feasible basis of any of the relaxed problems* (1.5), (1.6) *has one of the following structures, presented in terms of the subscripts:*

	$m+1$ *even*	$m+1$ *odd*
min problem	$\{0, i, i+1, ..., j, j+1, n\}$	$\{0, i, i+1, ..., j, j+1\}$
max problem	$\{0, 1, i, i+1, ..., j, j+1\}$ *or* $I_B \subset \{1, ..., n\}$	$\{0, 1, i, i+1, ..., j, j+1, n\}$ *or* $I_B \subset \{1, ..., n\}$

where I_B is the set of subscripts of the vectors that are in the basis B. In addition all dual feasible bases are dual nondegenerate, except for those with $I_B \subset \{1, ..., n\}$ which are dual degenerate.

Proof. We carry out the proof for the relaxed problem (1.5). The proof of the assertion for problem (1.6) is the same. For the sake of simplicity we prove the assertion for the case of $k_j = j,\ j = 1, ..., m$. The reasoning is, however applicable for the general case.

Let us write up in detailed form the matrix A of the equality constraints of problem (1.5), with the objective function coefficients on top of it:

	0	1	2	$\cdots$	$M-1$	M	$M+1$	$M+2$	$\cdots$	n
c^T	M	M	$M-1$	$\cdots$	2	1	1	2	$\cdots$	$n-M$
0	$M+1$	M	$M-1$	$\cdots$	2	1	1	2	$\cdots$	$n-M$
1	$0+\ldots+M$	$1+\ldots+M$	$2+\ldots+M$	$\cdots$	$M-1+M$	M	$M+1$	$M+1+M+2$	$\cdots$	$M+1+\ldots+n$
2	$\binom{0}{2}+\ldots+\binom{M}{2}$	$\binom{1}{2}+\ldots+\binom{M}{2}$	$\binom{2}{2}+\ldots+\binom{M}{2}$	$\cdots$	$\binom{M-1}{2}+\binom{M}{2}$	$\binom{M}{2}$	$\binom{M+1}{2}$	$\binom{M+1}{2}+\binom{M+2}{2}$	$\cdots$	$\binom{M+1}{2}+\ldots+\binom{n}{2}$
$\vdots$				$\vdots$					$\vdots$	
m	$\binom{0}{m}+\ldots+\binom{M}{m}$	$\binom{1}{m}+\ldots+\binom{M}{m}$	$\binom{2}{m}+\ldots+\binom{M}{m}$	$\cdots$	$\binom{M-1}{m}+\binom{M}{m}$	$\binom{M}{m}$	$\binom{M+1}{m}$	$\binom{M+1}{m}+\binom{M+2}{m}$	$\cdots$	$\binom{M+1}{m}+\ldots+\binom{n}{m}$

If $M=n$, then the columns below $M+1,...,n$ do not exist. A basis B in the minimization problem (1.5) is dual feasible if the following inequalities hold:

$$c_B^T B^{-1} a_p \le c_p \quad \text{for any nonbasic } p\ .$$

For the maximization problem the dual feasibility of a basis is defined by the reversed inequalities. A basis B is dual degenerate if there is at least one nonbasic p such that $c_p - c_B^T B^{-1} a_p = 0$. Since we have

$$\begin{pmatrix} 1 & c_B^T \\ 0 & B \end{pmatrix} \begin{pmatrix} c_p - c_B^T B^{-1} a_p \\ B^{-1} a_p \end{pmatrix} = \begin{pmatrix} c_p \\ a_p \end{pmatrix},$$

the first component of the solution of this equation can be expressed as

$$c_p - c_B^T B^{-1} a_p = \frac{1}{|B|} \begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}.$$

We are interested in the sign of $|B|$ and $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$. In connection with them we prove the following.

Lemma. We have the inequality $|B|>0$ and if $0 \in I_B$, then the determinant that comes out of $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$, if we put $\begin{pmatrix} c_p \\ a_p \end{pmatrix}$ in its right place (the column subscripts are in increasing order), is also positive, where p is a nonbasic subscript.

Proof of the Lemma. Since B is a basis, it follows that $|B| \ne 0$. We prove that this value is positive.

The entries in the first row can be written up as sum of 1's so that the number of terms in any position in that row is equal to the number of terms in any entry in its column. Then we apply a column subtraction procedure, further, split the obtained determinant into a sum of determinants. Any determinant in the obtained sum is either zero, or positive, by

Theorem 1, because they are minors, crossed out of the matrix P. At least one term must be positive because $|B| \neq 0$. It follows that $|B| > 0$.

Now we prove the second assertion. If $\begin{pmatrix} c_p \\ a_p \end{pmatrix}$ is put in its right place, then the first column of $\begin{pmatrix} c^T \\ A \end{pmatrix}$ will be the first column of the new determinant. If we subtract the first row from the second row in the determinant, then the first entry in the second row becomes -1 and the others 0. If we develop the determinant according to the second row, then, due to the special structure of the determinant, we obtain a minor of order $m+1$ crossed out of the matrix A. If $I_B = \{0, i_1, ..., i_m\}$, where $1 \le i_1 < ... < i_m$, then the subscript set of the columns of the minors is $\{i_1, ..., p, ..., i_m\}$, where $i_1 < ... < p < ... < i_m$. Thus, 0 is removed from I_B and p is included. It is not difficult to see that the positivity of $|B|$ implies the positivity of the minor. We have proved the Lemma.

Returning to the proof of the Theorem 2, consider first the case $0 \notin I_B$. Then $|B| > 0$ and

$$\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix} = \begin{cases} 0 & \text{if } p \neq 0 \\ < 0 & \text{if } p = 0 \ . \end{cases}$$

Hence, B is a dual feasible basis in the maximization problem.

If, on the other hand, $0 \in I_B$, then still $|B| > 0$ and by the Lemma, the determinant $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$ is equal to the $(m+1) \times (m+1)$ minor taken from A, corresponding to the columns $\{i_1, ..., p, ..., i_m\}$, multiplied by $(-1)^{h(p)}$, where $h(p)$ is the number of subscripts in B that are smaller than p. The minor is positive by the Lemma. We want to ensure the positivity of $\begin{vmatrix} c_p & c_B^T \\ a_p & B \end{vmatrix}$ for any nonbasic p. Now, if it is a minimization problem, then $h(p)$ must be even for any nonbasic p which implies that $\{i_1, ..., i_m\} = \{i, i+1, ..., j, j+1\}$ if m is even ($m+1$ is odd) and $\{i_1, ..., i_m\} = \{i, i+1, ..., j, j+1, n\}$ if m is odd ($m+1$ is even). If it is a maximization problem, then $h(p)$ must be odd for any nonbasic p which implies that $\{i_1, ..., i_m\} = \{1, i, i+1, ..., j, j+1, n\}$ if m is even ($m+1$ is odd) and $\{i_1, ..., i_m\} = \{1, i, i+1, ..., j, j+1\}$ if m is odd ($m+1$ is even). This proves the theorem.

Remark. If $k_j = j,\ j = 1, ..., m$, then all $(m+1) \times (m+1)$ submatrices of A are nonsingular. This is, however, not necessarily the case if $\{k_1, ..., k_m\} \neq \{1, ..., m\}$. Thus, when picking a dual feasible basis satisfying the structure in Theorem 2, we have to check on their independence as well.

3. Closed form bounds for the probability of the union based on S_{k_1}, S_{k_2}

Let $m = 2$ and assume that the binomial moments $S_{k_1}, S_{k_2},\ 1 \le k_1 < k_2 \le n$, are known. If $C(n,k) = \begin{pmatrix} n \\ k \end{pmatrix}$, then we have the following recurrence relation known as Pascal's rule:

$$C(n+1, k+1) = C(n,k) + C(n, k+1) \ .$$

By the use of these the coefficients of the equality constraints in the relaxed problems can be given as follows:

$$\sum_{s=i}^{j} \binom{s}{k} = \binom{j+1}{k+1} - \binom{i}{k+1} . \tag{3.1}$$

In view of (3.1) the relaxed version of problem (1.5) can be written in the form:

$$min(max) \ \{Mv_0 + \sum_{i=1}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n} (i-M)v_i\}$$

subject to

$$\sum_{i=0}^{M}(M-i+1)v_i + \sum_{i=M+1}^{n} (i-M)v_i = 1 \tag{3.2}$$

$$\sum_{i=0}^{M} \left[\binom{M+1}{k_1+1} - \binom{i}{k_1+1}\right]v_i + \sum_{i=M+1}^{n} \left[\binom{i+1}{k_1+1} - \binom{M+1}{k_1+1}\right]v_i = S_{k_1}$$

$$\sum_{i=0}^{M} \left[\binom{M+1}{k_2+1} - \binom{i}{k_2+1}\right]v_i + \sum_{i=M+1}^{n} \left[\binom{i+1}{k_2+1} - \binom{M+1}{k_2+1}\right]v_i = S_{k_2}$$

$$v_i \geq 0 \ , \ \ i = 0, ..., n \ .$$

Theorem 2 provides us with the following dual feasible bases for the above problem:

$$B_{min} = \{0, i, i+1\} \ , \ \ 1 \leq i \leq n-1 \ ,$$

$$B_{max} = \{0, 1, n\} \quad \text{or} \quad B_{max} \subset \{1, ..., n\} \ .$$

In order to present our formulas in compact forms we introduce the notations:

$$
\begin{aligned}
\Sigma^r_{i,j} &= (j-i+1)\binom{i-1}{r} - \binom{j+1}{r+1} + \binom{i}{r+1} \\
\Sigma^{r,t}_{i,j} &= \binom{i-1}{r}\left[\binom{j+1}{t+1} - \binom{i}{t+1}\right] - \binom{i-1}{t}\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] \\
\gamma^r_{i,j} &= i\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] - (j-i+1)\binom{i}{r+1} \\
\gamma^{r,t}_{i,j} &= \binom{i}{r+1}\left[\binom{j+1}{t+1} - \binom{i}{t+1}\right] - \binom{i}{t+1}\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] \\
\beta^r_{i,j} &= (j+1)\binom{j+1}{r} - \binom{j+1}{r+1} + \binom{i}{r+1} \\
\beta^{r,t}_{i,j} &= \binom{j+1}{r}\left[\binom{j+1}{t+1} - \binom{i}{t+1}\right] - \binom{j+1}{t}\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] \\
\alpha^r_{i,j} &= (j-i+1)\binom{j+1}{r} - \binom{j+1}{r+1} + \binom{i}{r+1} \\
\alpha^{r,t}_{i,j} &= \binom{j+1}{r}\left[\binom{j+1}{t+1} - \binom{i}{t+1}\right] - \binom{j+1}{t}\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] \\
\delta^r_{i,j} &= (i-1)\left[\binom{j+1}{r+1} - \binom{i}{r+1}\right] - (j-i)\binom{i}{r+1} .
\end{aligned}
\tag{3.3}
$$

We use problem (3.2) to present lower and upper bounds for $P(\nu \geq 1)$. To do this we find the optimal bases for the minimization and maximization problems, respectively. We already have a full description of the dual feasible bases. What we need is to find those (one for the min problem and one for the max problem) that are also primal feasible. Three cases will be considered.

Case 1. Let $1 \leq i \leq M-1$. The primal feasibility conditions for B_{min} are given below:

$$
S_{k_1}\Sigma^{k_2}_{i+1,M} - S_{k_2}\Sigma^{k_1}_{i+1,M} + \Sigma^{k_1,k_2}_{i+1,M} \geq 0 ,
$$

$$
S_{k_1}\gamma^{k_2}_{i+1,M} - S_{k_2}\gamma^{k_1}_{i+1,M} - \gamma^{k_1,k_2}_{i+1,M} \geq 0 ,
$$

$$
S_{k_1}\gamma^{k_2}_{i,M} - S_{k_2}\gamma^{k_1}_{i,M} + \gamma^{k_1,k_2}_{i,M} \leq 0 .
$$

In this case the closed form lower bound for $P(\nu \geq 1)$ is expressed by

$$
1 - \frac{S_{k_1}\Sigma^{k_2}_{i+1,M} - S_{k_2}\Sigma^{k_1}_{i+1,M} + \Sigma^{k_1,k_2}_{i+1,k}}{i\,\Sigma^{k_1,k_2}_{i+1,M} + \binom{i}{k_1+1}\Sigma^{k_2}_{i+1,M} - \binom{i}{k_2+1}\Sigma^{k_1}_{i+1,M}} \leq P(\nu \geq 1) , \tag{3.4}
$$

where $\Sigma^r_{i,j}, \Sigma^{r,t}_{i,j}, \gamma^r_{i,j}, \gamma^{r,t}_{i,j}$ are given in (3.3).

Case 2. Let $i = M$. The conditions that ensure the primal feasibility of $B_{min} = \{0, M, M+1\}$ are as follows:

$$
S_{k_1}\binom{M}{k_2-1} - S_{k_2}\binom{M}{k_1-1} - \frac{k_2-k_1}{M-k_2+1}\binom{M+1}{k_1}\binom{M}{k_2} \geq 0 ,
$$

$$S_{k_1}\beta^{k_2}_{1,M} - S_{k_2}\beta^{k_1}_{1,M} + \beta^{k_1,k_2}_{1,M} \geq 0 \ ,$$

$$S_{k_1}\beta^{k_2}_{1,M-1} - S_{k_2}\beta^{k_1}_{1,M-1} + \beta^{k_1,k_2}_{1,M-1} \geq 0 \ .$$

The corresponding closed form lower bound for $P(\nu \geq 1)$ is given by

$$1 \ - \ \frac{S_{k_1}\binom{M}{k_2-1} - S_{k_2}\binom{M}{k_1-1} \ - \ \frac{k_2-k_1}{M-k_2+1}\binom{M+1}{k_1}\binom{M}{k_2}}{\beta^{k_1,k_2}_{1,M-1} \ - \ \frac{M\ (k_2-k_1)}{M-k_2+1}\binom{M+1}{k_1}\binom{M}{k_2}} \ \leq \ P(\nu \geq 1) \ , \qquad (3.5)$$

where $\beta^{r}_{i,j}, \beta^{r,t}_{i,j}$ are given in (3.3).

Case 3. Let $M+1 \leq i \leq n-1$. B_{min} is primal feasible if and only if i is determined by the following conditions:

$$S_{k_1}\alpha^{k_2}_{M+1,i} - S_{k_2}\alpha^{k_1}_{M+1,i} + \alpha^{k_1,k_2}_{M+1,i} \geq 0 \ ,$$

$$S_{k_1}\gamma^{k_2}_{M+1,i+1} - S_{k_2}\gamma^{k_1}_{M+1,i+1} - \gamma^{k_1,k_2}_{M+1,i+1} \geq 0 \ ,$$

$$S_{k_1}\gamma^{k_2}_{M+1,i} - S_{k_2}\gamma^{k_1}_{M+1,i} - \gamma^{k_1,k_2}_{M+1,i} \geq 0 \ .$$

Then the closed form lower bound is the following:

$$1 \ - \ \frac{S_{k_1}\alpha^{k_2}_{M+1,i} - S_{k_2}\alpha^{k_1}_{M+1,i} + \alpha^{k_1,k_2}_{M+1,i}}{\binom{M+1}{k_1+1}\alpha^{k_2}_{M+1,i} - \binom{M+1}{k_2+1}\alpha^{k_1}_{M+1,i} + \ (M+1)\ \alpha^{k_1,k_2}_{M+1,i}} \ \leq \ P(\nu \geq 1) \ , \qquad (3.6)$$

where $\alpha^{r}_{i,j}, \alpha^{r,t}_{i,j}, \gamma^{r}_{i,j}, \gamma^{r,t}_{i,j}$ are given in (3.3).

If $B_{max} \subset \{1,...,n\}$ is primal feasible in the relaxed version of the maximization problem (3.2), then the upper bound for the probability of the union is equal to 1.

The basis $B_{max} = \{0,1,n\}$ is primal feasible if and only if the following conditions hold:

$$S_{k_1}\delta^{k_2}_{M+1,n} - S_{k_2}\delta^{k_1}_{M+1,n} - \gamma^{k_1,k_2}_{M+1,n} \leq 0 \ ,$$

$$S_{k_1}\gamma^{k_2}_{M+1,n} - S_{k_2}\gamma^{k_1}_{M+1,n} - \gamma^{k_1,k_2}_{M+1,n} \geq 0 \ ,$$

$$S_{k_1}\binom{M+1}{k_2+1} \ \leq \ S_{k_2}\binom{M+1}{k_1+1} \ .$$

The corresponding closed form upper bound for $P(\nu \geq 1)$ is given below:

$$P(\nu \geq 1) \ \leq \ \frac{S_{k_1}\delta^{k_2}_{M+1,n} - S_{k_2}\delta^{k_1}_{M+1,n}}{\gamma^{k_1,k_2}_{M+1,n}} \ , \qquad (3.7)$$

where $\delta^{r}_{i,j}, \gamma^{r}_{i,j}, \gamma^{r,t}_{i,j}$ are given in (3.3).

If we use the relaxed version of problem (1.6), rather than that of problem (1.5), then the lower and upper bounds change in such a way that we have to replace $M-1$ for M

in the formulas of Section 3.

4. Closed form bounds for the probability of the union based on S_1, S_2, S_3

We look at the relaxed versions of problems (1.5), (1.6) and create bounds for the probability of the union, based on the knowledge of the binomial moments S_1, S_2, S_3. Since $m+1$ is even, then by the use of Theorem 2, we derive that any dual feasible basis B_{min} of the relaxed version of the minimization problem (1.5) has the form:

$$B_{min} = \{0, i, i+1, n\} \ , \quad i = 1, ..., n-2 \ .$$

Similarly, any dual feasible basis B_{max} of relaxed version of the maximization problem has the form:

$$B_{max} = \{0, 1, i, i+1\}, \ i = 2, ..., n-1, \quad \text{or} \quad B_{max} \subset \{1, ..., n\} \ .$$

Below we present conditions that ensure the primal feasibility of B_{min} as well as the corresponding lower bounds for $P(\nu \geq 1)$, i.e., the probability of the union of the events.
Case 1. Let $1 \leq i \leq M-1$. B_{min} is primal feasible if and only if i is determined by the conditions

$$2[iM + (n-1)(i+M-1)]S_1 - 6(n+i+M-3)S_2 + 24S_3 \geq Mni \ ,$$

$$2[M(i-1) + (n-1)(i+M-2)]S_1 - 6(n+i+M-4)S_2 + 24S_3 \leq Mn(i-1) \ ,$$

$$2(i-1)(i+2M-2)S_1 - 6(2i+M-4)S_2 + 24S_3 \geq Mi(i-1) \ ,$$

$$\begin{aligned} &2[i(n+2M+i) + (n-1)(i+M-1)]S_1 - 6(n+2i+M-3)S_2 + 24S_3 \\ &\qquad \leq \ i[M(2n+i+1) + (i+1)(n+1)] \ . \end{aligned}$$

In this case the lower bound for $P(\nu \geq 1)$ is obtained as follows:

$$\frac{2[i(n+2M+i) + (n-1)(i+M-1)]S_1 - 6(n+2i+M-3)S_2 + 24S_3}{(n+1)(M+1)(i+1)i} + \frac{Mn(i-1)}{(n+1)(M+1)(i+1)} \ \leq \ P(\nu \geq 1) \ . \tag{4.1}$$

Case 2. Let $i = M$. Basis $B_{min} = \{0, M, M+1, n\}$ is primal feasible if and only if the following conditions are satisfied:

$$2M(2n+M-1)S_1 - 6(n+2M-2)S_2 + 24S_3 \geq M(M+1)n \ ,$$

$$2(M-1)(2n+M-2)S_1 - 6(n+2M-4)S_2 + 24S_3 \leq (M-1)Mn \ ,$$

$$6M(M-1)S_1 - 18S_2 + 24S_3 \geq (M-1)M(M+1) \ ,$$

$$6M(n+M)S_1 - 6(n+3M-2)S_2 + 24S_3 \le (M+1)(3n+M+2)\ .$$

The corresponding lower bound for $P(\nu \ge 1)$ is given below:

$$\frac{6M(n+M)S_1 - 6(n+3M-2)S_2 + 24S_3}{M(M+1)(M+2)(n+1)} + \frac{n(M-1)}{(M+2)(n+1)} \ \le\ P(\nu \ge 1)\ . \qquad (4.2)$$

Case 3. Let $M+1 \le i \le n-2$. B_{min} is primal feasible if and only if i satisfies the following conditions:

$$2[nM + i(n+M-1)]S_1 - 6(n+i+M-2)S_2 + 24S_3 \ge Mn(i+1)\ ,$$

$$2[iM + (n-1)(i+M-1)]S_1 - 6(n+i+M-3)S_2 + 24S_3 \le Mni\ ,$$

$$2i(i+2M-1)S_1 - 6(2i+M-2)S_2 + 24S_3 \ge i(M+1)M\ ,$$

$$2[i(n+2M+i) + (n+1)(i+M+1)]S_1 - 6(n+2i+M-1)S_2 + 24S_3$$
$$\le\ (i+1)[iM + (n+1)(i+2M+2)]\ .$$

In this case the lower bound is obtained as follows:

$$\frac{2[i(n+2M+i) + (n+1)(i+M+1)]S_1 - 6(n+2i+M-1)S_2 + 24S_3}{(i+1)(i+2)(M+1)(n+1)}$$
$$+\frac{niM}{(i+2)(M+1)(n+1)} \ \le\ P(\nu \ge 1)\ . \qquad (4.3)$$

In order to obtain an upper bound for $P(\nu \ge 1)$ we consider the relaxed version of the maximization problem (1.5). Note that if the dual feasible basis $B_{max} \subset \{1, ..., n\}$ is also primal feasible, then the optimum value of the maximization problem, i.e., the upper bound for the probability of the union, is equal to 1. As before, we have three cases for the choice of i.

Case 1. Let $2 \le i \le M-1$. The primal feasibility conditions for the basis $B_{max} = \{0, 1, i, i+1\}$ are as follows:

$$2(i-1)(i+2M-2)S_1 - 6(2i+M-4)S_2 + 24S_3 \ge M(i-1)i\ ,$$

$$2(i-1)(M-1)S_1 - 6(i+M-3)S_2 + 24S_3 \le 0\ ,$$

$$2(i-2)(M-1)S_1 - 6(i+M-4)S_2 + 24S_3 \ge 0\ ,$$

$$2[i(i+M) + (i-1)(M-1)]S_1 - 6(2i+M-3)S_2 + 24S_3 \le i(i+1)(M+1)\ .$$

The corresponding upper bound for $P(\nu \ge 1)$ is presented below:

$$P(\nu \ge 1) \ \le\ \frac{2[i(i+M) + (i-1)(M-1)]S_1 - 6(2i+M-3)S_2 + 24S_3}{i(i+1)(M+1)}\ . \qquad (4.4)$$

Case 2. Let $i = M$. The basis $B_{max} = \{0, 1, M, M+1\}$ is primal feasible if and only if

$$6M(M-1)S_1 - 18(M-1)S_2 + 24S_3 \geq (M-1)M(M+1) ,$$

$$2M(M-1)S_1 - 12(M-1)S_2 + 24S_3 \leq 0 ,$$

$$2(M-1)(M-2)S_1 - 12(M-2)S_2 + 24S_3 \geq 0 ,$$

$$6M^2S_1 - 6(3M-2)S_2 + 24S_3 \leq M(M-1)(M+1) .$$

The corresponding upper bound for $P(\nu \geq 1)$ is given below:

$$P(\nu \geq 1) \ \leq \ \frac{6M^2S_1 - 6(3M-2)S_2 + 24S_3}{M(M+1)(M+2)} . \tag{4.5}$$

Case 3. Let $M+1 \leq i \leq n-1$. The basis B_{max} is primal feasible if and only if i is determined by the following conditions:

$$2i(i+2M-1)S_1 - 6(2i+M-2)S_2 + 24S_3 \geq i(i+1)M ,$$

$$2i(M-1)S_1 - 6(i+M-2)S_2 + 24S_3 \leq 0 ,$$

$$2(i-1)(M-1)S_1 - 6(i+M-3)S_2 + 24S_3 \geq 0 ,$$

$$2[i(i+M) + (i+1)(M+1)]S_1 - 6(2i+M-1)S_2 + 24S_3 \leq (i+1)(i+2)(M+1) .$$

With i satisfying these inequalities we have the upper bound given by:

$$P(\nu \geq 1) \ \leq \ \frac{2[i(i+M) + (i+1)(M+1)]S_1 - 6(2i+M-1)S_2 + 24S_3}{(i+1)(i+2)(M+1)} . \tag{4.6}$$

If we replace $M-1$ for M in the formulas of Section 4, then we obtain the closed form bounds that come out of the relaxed version of problem (1.6).

5. Closed form upper bounds for the probability of the union based on S_1, S_2, S_3, S_4

In this section we present upper bound formulas for the probability of the union of events based on the first four binomial moments.

Since $m+1$ is odd, then by Theorem 2, any dual feasible basis B_{max} of the relaxed version of the maximization problem (1.5) or (1.6) is of the form:

$$B_{max} = \{0, 1, i, i+1, n\}, \ i = 2, ..., n-1, \quad \text{or} \quad B_{max} \subset \{1, ..., n\} .$$

If $B_{max} \subset \{1, ..., n\}$, then the upper bound for $P(\nu \geq 1)$ is 1. In order to determine the index i that ensures the primal feasibility of the basis of the form $B_{max} = \{0, 1, i, i+1, n\}$ we consider the following cases.

Case 1. Let $2 \leq i \leq M-1$. The primal feasibility conditions are:

$$2[i(i-1)(n+M)-(M-1)(n-1)+2i(nM+1)]S_1$$
$$-6(3n+3M-nM-i^2+5i-2iM-2ni-7)S_2+24(n+2i+M-6)S_3-120S_4$$
$$\leq\ i(i+1)(n+1)(M+1)\ ,$$
$$2(i-1)(ni+iM+2nM-2n-i-2M+2)S_1-6[nM+(i-2)(2n+2M+i-5)]S_2$$
$$+24(n+2i+M-7)S_3-120S_4\ \geq\ (i-1)inM\ ,$$
$$2(M-1)(n-1)(i-1)S_1-6(ni+iM+nM-3n-3i-3M+7)S_2$$
$$+24(n+i+M-6)S_3-120S_4\ \leq\ 0\ ,$$
$$2(M-1)(n-1)(i-2)S_1-6(ni+iM+nM-4n-3i-4M+10)S_2$$
$$+24(n+i+M-7)S_3-120S_4\ \geq\ 0\ ,$$
$$2(M-1)(i-1)(i-2)S_1-6(i-2)(i+2M-5)S_2$$
$$+24(2i+M-7)S_3-120S_4\ \leq\ 0\ .$$

Under these conditions the corresponding upper bound is given below:

$$\begin{aligned} P(\nu\geq 1)\ \leq\ & \frac{2[i(i-1)(n+M)-(M-1)(n-1)+2i(nM+1)]S_1}{(M+1)i(i+1)(n+1)} \\ & -\ \frac{6(3n+3M-nM-i^2+5i-2iM-2ni-7)S_2}{(M+1)i(i+1)(n+1)} \\ & +\ \frac{24(n+2i+M-6)S_3-120S_4}{(M+1)i(i+1)(n+1)}\ . \end{aligned} \tag{5.1}$$

Case 2. Let $i = M$. The basis $B_{max} = \{0, 1, M, M+1, n\}$ is primal feasible if and only if

$$2M(3nM+M^2+2)S_1-6(3M^2+(3M-2)(n-2))S_2+24(n+3M-5)S_3-120S_4$$
$$\leq\ M(M+1)(M+2)(n+1)\ ,$$
$$2M(M-1)(3n+M-2)S_1-18(M-1)(n+M-2)S_2+24(n+3M-6)S_3-120S_4$$
$$\geq\ M(M-1)(M+1)n\ ,$$
$$2M(M-1)(n-1)S_1-6(M-1)(2n+M-4)S_2+24(n+2M-5)S_3-120S_4\ \leq\ 0\ ,$$
$$2(M-1)(M-2)(n-1)S_1-6(M-2)(2n+M-5)S_2+24(n+2M-7)S_3-120S_4\ \geq\ 0\ ,$$
$$2M(M-1)(M-2)S_1-18(M-1)(M-2)S_2+72(M-2)S_3-120S_4\ \leq\ 0\ .$$

The closed form upper bound for $P(\nu \geq 1)$ is given by

$$P(\nu \geq 1) \;\leq\; \frac{2M(3nM + M^2 + 2)S_1 - 6(3M^2 + (3M-2)(n-2))S_2}{M(M+1)(M+2)(n+1)} + \frac{24(n + 3M - 5)S_3 - 120S_4}{M(M+1)(M+2)(n+1)} \,. \tag{5.2}$$

Case 3. Let $M+1 \leq i \leq n-2$. B_{max} is primal feasible if and only if i satisfies the conditions:

$$2[(i+1)(ni+nM+iM+1)+n+i+M]S_1 - 6[(i-1)(i-2)+2i(n+M)+(n-1)(M-1)]S_2$$
$$+24(n+2i+M-4)S_3 - 120S_4 \;\leq\; (i+1)(i+2)(M+1)(n+1) \,,$$
$$2i[n(i+M)+(M-1)(n+i-1)]S_1 - 6[(i-1)(2n+2M+i-4)+nM]S_2$$
$$+24(n+2i+M-5)S_3 - 120S_4 \;\geq\; ni(i+1)M \,,$$
$$2i(M-1)(n-1)S_1 - 6[(n-2)(i+M-2)+i(M-1)]S_2$$
$$+24(n+i+M-5)S_3 - 120S_4 \;\leq\; 0 \,,$$
$$2(i-1)(M-1)(n-1)S_1 - 6[(n-3)(i+M-1)+iM-2n+4]S_2$$
$$+24(n_i+M-6)S_3 - 120S_4 \;\geq\; 0 \,,$$
$$2i(i-1)(M-1)S_1 - 6(i-1)(i+2M-4)S_2 + 24(2i+M-5)S_3 - 120S_4 \;\leq\; 0 \,.$$

The corresponding upper bound for $P(\nu \geq 1)$ is given by

$$P(\nu \geq 1) \;\leq\; \frac{2[(i+1)(ni+nM+iM+1)+n+i+M]S_1}{(i+1)(i+2)(n+1)(M+1)} - \frac{6[(i-1)(i-2)+2i(n+M)+(n-1)(M-1)]S_2}{(i+1)(i+2)(n+1)(M+1)} + \frac{24(n+2i+M-4)S_3 - 120S_4}{(i+1)(i+2)(n+1)(M+1)} \,. \tag{5.3}$$

As before, if we apply our bounding technique on the relaxed problem (1.6), rather than (1.5), then the just derived formulas provide us with the upper bounds if we replace $M-1$ for M.

6. Algorithmic bounds

In Sections 3, 4 and 5 we have derived closed form bounds for the probability of the union, by the use of the relaxed problems (1.5), (1.6) for the cases of $m = 2,3,4$. For larger m values the solution of the relaxed problems can be obtained by specially designed dual algorithms of linear programming. Once an algorithm of this kind terminates, the solutions for the non-relaxed problem can be continued again by the dual algorithm. In fact, as it is well known in linear programming, the dual algorithm can efficiently be used, as a reoptimization technique, whenever the optimal basis has already been found but a further constraint is introduced into the problem.

The algorithm presented below works in this way and is applicable to cases with consecutive and non-consecutive moments. We remark that it is more practical to carry out the algorithms to obtain the bound, rather than to apply a complicated closed form formula.

***Algorithmic solutions of problems* (1.5), (1.6)**

Step 0. Find an initial dual feasible basis B to the relaxed problem. Any basis that has the structure presented in Theorem 2 is suitable.

Step 1. Check for primal feasibility. If $B^{-1}b \geq 0$, then the solution of the relaxed problem terminates. Go to Step 4. Otherwise go to Step 2.

Step 2. If $(B^{-1}b)_j < 0$, then the jth vector in B (not necessarily equal to a_j) is a candidate to leave the basis. Choose arbitrarily among the candidates to leave the basis. Go to Step 3.

Step 3. Include the vector a_l into the basis that restores the dual feasible basis structure. Go to Step 1.

Step 4. If the additional constraint $v_0 + ... + v_M \geq v_{M+1} + ... + v_n$ (or $v_M + ... + v_n \geq v_0 + ... + v_{M-1}$) is satisfied, then the solution of problem (1.5) (or (1.6)) terminates. Otherwise go to Step 5.

Step 5. Reoptimize the problem with the additional constraint (1.5a) or (1.6a): introduce slack variable into the additional inequality constraint, prescribe nonnegativity relation for the slack variable, set up the new dual tableau and carry out the dual method.

If the sequence of probabilities $p_0, ..., p_n$ is increasing or decreasing, i.e., if $M = n$ or $M = 0$, then the solution of problem (1.5) or (1.6) terminates with Step 3. No reoptimization is needed. The relaxed problem is equivalent to the original problem (1.5) or (1.6).

7. Application in Reliability

Let $A_1, ..., A_n$ be independent events and define the random variables $X_1, ..., X_n$ as the characteristic variables corresponding to the above events, respectively, i.e.,

$$X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs ,} \\ 0 & \text{otherwise .} \end{cases}$$

Let $p_i = P(X_i = 1),\ i = 1, ..., n$. The random variables $X_1, ..., X_n$ have logconcave discrete distributions. Since the convolution of discrete logconcave sequences is logconcave (see, e.g., Prékopa [11]), it follows that the distribution of $X_1 + ... + X_n$ is also logconcave.

In many applications it is an important problem to compute, or at least approximate, e.g., by the use of bounds, the probability

$$P(X_1 + ... + X_n \geq 1) \ . \tag{7.1}$$

If $I_1, ..., I_{C(n,k)}$ designate the *k*-element subsets of the set $\{1, ..., n\}$ and $J_l = \{1, ..., n\} \backslash I_l$, $l = 1, ..., C(n, k)$, then we have the equation

$$P(X_1 + ... + X_n \geq 1) = \sum_{k=1}^{n} \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i \prod_{j \in J_l} (1 - p_j) \ , \tag{7.2}$$

where $C(n, k) = \binom{n}{k}$.

If n is large, then the calculation of the probabilities on the right hand side of (7.2) may be hard, even impossible. However, we can calculate lower and upper bounds for the probability on the left hand side of (7.2) by the use of the sums:

$$S_k = \sum_{1 \leq i_1 < ... < i_k \leq n} p_{i_1} ... p_{i_k} = \sum_{l=1}^{C(n,k)} \prod_{i \in I_l} p_i \ , \quad k = 1, ..., m \ , \tag{7.3}$$

where m may be much smaller than n. Since the random variable $X_1 + ... + X_n$ has logconcave, hence unimodal distribution we can impose the unimodality condition on the probability distribution:

$$P(X_1 + ... + X_n = k) \ , \quad k = 0, ..., n \ . \tag{7.4}$$

Then we solve both the minimization and maximization problems presented in Section 1, to obtain the bounds for the probability (7.1). If m is small, then the bounds can be obtained by the formulas of Section 3, 4 and 5. Note that the largest probability (7.4) corresponds to

$$k_{max} = \left\lfloor (n + 1) \frac{p_1 + ... + p_n}{n} \right\rfloor .$$

The inclusion-exclusion formula provides us with the probability (7.1), in terms of the binomial moments $S_1, ..., S_n$:

$$P(X_1 + ... + X_n \geq 1) = \sum_{k=1}^{n} (-1)^{k-1} S_k \ . \tag{7.5}$$

However, to compute higher order binomial moments may be extremely difficult, sometimes impossible. The advantage of our approach is that we use the first few binomial moments $S_1, ..., S_m$, where m is relatively small and in many cases we can obtain very good bounds.

8. Numerical examples

We present numerical examples to show that if the probability distribution is unimodal with known mode, M, then by the use of our bounding methodology, we can obtain tighter bounds for the probability of the union. In the following examples LB and UB stand for lower and upper bounds, respectively.

Example 1. We assume that the first m binomial moments of the events are known. In Table 1 we present bounds for the probability of the union with and without the unimodality condition.

		with unimodality						without unimodality					
		$m=2$		$m=3$		$m=4$		$m=2$		$m=3$		$m=4$	
n	M	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
10	3	0.9172	0.9404	0.9205	0.9326	0.9227	0.9281	0.7317	1	0.8190	1	0.8662	1
10	5	0.9255	0.9562	0.9319	0.9470	0.9334	0.9401	0.7594	1	0.8423	1	0.8815	1
10	7	0.9310	0.9735	0.9389	0.9603	0.9403	0.9475	0.7832	1	0.8580	1	0.8923	1
10	3	0.9093	0.9690	0.9151	0.9380	0.9173	0.9280	0.7225	1	0.8129	1	0.8618	1
10	5	0.9248	0.9753	0.9312	0.9544	0.9341	0.9447	0.7617	1	0.8442	1	0.8843	1
10	7	0.9339	0.9884	0.9424	0.9687	0.9446	0.9540	0.7935	1	0.8662	1	0.8998	1
10	3	0.9096	0.9446	0.9128	0.9259	0.9140	0.9201	0.7197	1	0.8078	1	0.8559	1
10	5	0.9187	0.9483	0.9225	0.9361	0.9244	0.9308	0.7428	1	0.8267	1	0.8697	1
10	7	0.9241	0.9570	0.9296	0.9459	0.9311	0.9371	0.7622	1	0.8405	1	0.8787	1
10	3	0.9114	0.9516	0.9156	0.9322	0.9174	0.9255	0.7236	1	0.8126	1	0.8609	1
10	5	0.9230	0.9622	0.9286	0.9470	0.9309	0.9394	0.7544	1	0.8381	1	0.8790	1
10	7	0.9300	0.9755	0.9375	0.9600	0.9394	0.9473	0.7812	1	0.8563	1	0.8911	1
10	3	0.9046	0.9364	0.9074	0.9185	0.9084	0.9132	0.7099	1	0.7990	1	0.8480	1
10	5	0.9132	0.9367	0.9161	0.9276	0.9176	0.9227	0.7301	1	0.8144	1	0.8604	1
10	7	0.9185	0.9447	0.9227	0.9352	0.9185	0.9286	0.7456	1	0.8270	1	0.7456	1

Table 1

The bounds for $P(\nu \geq 1)$, obtained by the use of the relaxed problems (1.5) and (1.6), are presented in Table 2.

		Relaxed Problem (1.5)						Relaxed Problem (1.6)					
		$m=2$		$m=3$		$m=4$		$m=2$		$m=3$		$m=4$	
n	M	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB
10	3	0.9156	0.9404	0.9205	0.9326	0.9227	0.9288	0.9172	0.9460	0.9205	0.9333	0.9227	0.9281
10	5	0.9238	0.9592	0.9319	0.9500	0.9334	0.9401	0.9255	0.9562	0.9315	0.9470	0.9334	0.9401
10	7	0.9299	0.9805	0.9386	0.9641	0.9401	0.9475	0.9310	0.9735	0.9389	0.9603	0.9403	0.9477
10	3	0.9086	0.9690	0.9139	0.9380	0.9173	0.9280	0.9093	0.9986	0.9151	0.9455	0.9173	0.9280
10	5	0.9230	0.9753	0.9312	0.9567	0.9341	0.9450	0.9248	0.9788	0.9312	0.9544	0.9339	0.9447
10	7	0.9328	0.9949	0.9424	0.9731	0.9444	0.9540	0.9339	0.9884	0.9422	0.9687	0.9446	0.9551
10	3	0.9091	0.9446	0.9120	0.9259	0.9140	0.9201	0.9096	0.9618	0.9128	0.9300	0.9140	0.9201
10	5	0.9176	0.9483	0.9225	0.9376	0.9244	0.9310	0.9187	0.9500	0.9225	0.9361	0.9242	0.9308
10	7	0.9232	0.9609	0.9296	0.9485	0.9309	0.9371	0.9241	0.9570	0.9295	0.9459	0.9311	0.9377
10	3	0.9105	0.9516	0.9147	0.9322	0.9174	0.9256	0.9114	0.9700	0.9156	0.9368	0.9174	0.9255
10	5	0.9214	0.9622	0.9286	0.9491	0.9309	0.9394	0.9230	0.9635	0.9285	0.9470	0.9307	0.9394
10	7	0.9290	0.9811	0.9375	0.9636	0.9392	0.9473	0.9300	0.9755	0.9375	0.9600	0.9394	0.9480
10	3	0.9046	0.9364	0.9069	0.9185	0.9084	0.9132	0.9042	0.9562	0.9074	0.9243	0.9084	0.9140
10	5	0.9126	0.9367	0.9161	0.9277	0.9176	0.9230	0.9132	0.9412	0.9161	0.9276	0.9175	0.9227
10	7	0.9179	0.9459	0.9227	0.9368	0.9238	0.9286	0.9185	0.9447	0.9225	0.9352	0.9238	0.9292

Table 2

Example 2. In this example we assume that two (not necessarily consecutive) binomial moments, $S_{k_1},\ S_{k_2}$, $(1 \le k_1 < k_2 \le n)$, are known. In Table 2 and 3 we present bounds for $P(\nu \ge 1)$ with and without the unimodality condition, respectively.

		with unimodality									
		k_1=1 k_2=3		k_1=1 k_2=4		k_1=2 k_2=3		k_1=2 k_2=4		k_1=3 k_2=4	
n	M	*LB*	*UB*	LB	UB	LB	UB	LB	UB	LB	UB
10	3	0.9160	0.9480	0.9160	0.9540	0.9140	0.9970	0.9130	1	0.9120	1
10	5	0.9240	0.9650	0.9240	0.9700	0.9220	0.9970	0.9210	1	0.9200	1
10	7	0.9300	0.9840	0.9290	0.9900	0.9270	1	0.9260	1	0.9240	1
10	3	0.9070	1	0.9060	1	0.9030	1	0.9020	1	0.9000	1
10	5	0.9230	0.9910	0.9220	1	0.9200	1	0.9190	1	0.9170	1
10	7	0.9320	1	0.9310	1	0.9290	1	0.9280	1	0.9260	1
10	3	0.9080	0.9630	0.9080	0.9800	0.9060	1	0.9050	1	0.9040	1
10	5	0.9180	0.9570	0.9170	0.9640	0.9160	0.9930	0.9150	1	0.9140	1
10	7	0.9230	0.9650	0.9220	0.9720	0.9210	0.9960	0.9200	1	0.9190	1
10	3	0.9100	0.9710	0.9090	0.9880	0.9070	1	0.9060	1	0.9040	1
10	5	0.9220	0.9730	0.9210	0.9810	0.9190	1	0.9180	1	0.9170	1
10	7	0.9290	0.9870	0.9280	0.9960	0.9260	1	0.9250	1	0.9230	1
10	3	0.9040	0.9540	0.9030	0.9710	0.9020	1	0.9020	1	0.9000	1
10	5	0.9130	0.9440	0.9120	0.9510	0.9110	0.9740	0.9100	0.9920	0.9090	1
10	7	0.9180	0.9520	0.9170	0.9570	0.9160	0.9750	0.9160	0.9830	0.9150	1

Table 3

		without unimodality									
		k_1=1 k_2=3		k_1=1 k_2=4		k_1=2 k_2=3		k_1=2 k_2=4		k_1=3 k_2=4	
n	*M*	*LB*	*UB*	LB	UB	LB	UB	LB	UB	LB	UB
10	3	0.6850	1	0.6480	1	0.5470	1	0.5080	1	0.4330	1
10	5	0.7150	1	0.6780	1	0.5830	1	0.5440	1	0.4670	1
10	7	0.7410	1	0.7060	1	0.6210	1	0.5840	1	0.5100	1
10	3	0.6680	1	0.6320	1	0.5270	1	0.4790	1	0.4030	1
10	5	0.7150	1	0.6790	1	0.5830	1	0.5350	1	0.4570	1
10	7	0.7530	1	0.7160	1	0.6320	1	0.5950	1	0.5200	1
10	3	0.6690	1	0.6330	1	0.5320	1	0.4880	1	0.4140	1
10	5	0.6970	1	0.6610	1	0.5640	1	0.5220	1	0.4460	1
10	7	0.7200	1	0.6830	1	0.5930	1	0.5570	1	0.4830	1
10	3	0.6730	1	0.6360	1	0.5340	1	0.4900	1	0.4150	1
10	5	0.7090	1	0.6720	1	0.5760	1	0.5330	1	0.4550	1
10	7	0.7390	1	0.7030	1	0.6170	1	0.5790	1	0.5050	1
10	3	0.6590	1	0.6230	1	0.5210	1	0.4780	1	0.4050	1
10	5	0.6820	1	0.6460	1	0.5480	1	0.5040	1	0.4300	1
10	7	0.7020	1	0.6660	1	0.5720	1	0.5340	1	0.4610	1

Table 4

Example 3. We give an illustration of the algorithm that we have presented in Section 6. Assume that the probability distribution is unimodal and its mode is 5. Let $n = 10$, $S_1 = 5.3568245$, $S_2 = 16.2332237$, $S_3 = 32.377332$.

We consider the relaxed version of the minimization problem (1.6) and choose the initial basis $B = \{0, 2, 3, 10\}$, which is dual feasible by Theorem 2.

***Iteration* 1**

Step 0. Initial dual feasible basis: $B = \{0, 2, 3, 10\}$. **Step 1.** Since

$$B^{-1}b = \begin{pmatrix} 0.070527273 \\ -0.024297844 \\ 0.06646205 \\ 0.097888845 \end{pmatrix} \ngeq 0 ,$$

it follows that B is not primal feasible.

Step 2. The second vector in B, that is a_2, leaves the basis since $(B^{-1}b)_2 < 0$.

Step 3. The vector a_4 restores the dual feasible basis structure, hence it enters the basis.

We proceed to the second iteration with the updated basis, $B = \{0, 4, 3, 10\}$.

***Iteration* 2**

Step 1. We have

$$B^{-1}b = \begin{pmatrix} 0.068539267 \\ 0.046860129 \\ 0.0117919 \\ 0.097809956 \end{pmatrix} > 0 \; .$$

Thus B is optimal and the optimum value of the relaxed problem (1.6) is 0.931460733. The solution of the relaxed problem terminates.

Step 4. The additional constraint (1.6a) is equivalent to

$$v_5 + ... + v_{10} - v_0 - ... - v_4 = -0.02938134 \; < \; 0 \; .$$

The optimal solution to the relaxed problem does not satisfy constraint (1.6a).

Step 5. In order to ensure the mode of the distribution is 5 we prescribe (1.6a) as an additional constraint:

$$v_5 + ... + v_{10} - v_0 - ... - v_4 \; \geq \; 0 \; .$$

Let us rewrite the constraint in the form

$$v_5 + ... + v_{10} - v_0 - ... - v_4 - v_{11} \; = \; 0 \; ,$$

where $v_{11} \geq 0$ is slack variable. We use the dual method to reoptimize the problem (see, e.g., [12]) After applying the dual method to the new problem, we obtain the optimal basis and the optimum value of problem (1.6), i.e., the lower bound for the probability of the union as given below:

$$\begin{pmatrix} v_0 \\ v_3 \\ v_4 \\ v_5 \\ v_{10} \end{pmatrix} = \begin{pmatrix} 0.0685393 \\ 0.0117919 \\ 0.0468601 \\ 0.0097938 \\ 0.09780996 \end{pmatrix} \quad \text{and} \quad 0.931905905 \; \leq \; P(\nu \geq 1) \; .$$

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