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SPECIAL ISSUE ON COMPLEX ANALYSIS: THEORY AND APPLICATIONS DEDICATED TO PROFESSOR HARI M. SRIVASTAVA, ON THE OCCASION OF HIS 70TH BIRTHDAY

Coefficient Estimate for a Subclass of Univalent Functions with Respect to Symmetric Points

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Abstract. In this paper, the subclasses $\mathscr{S}_s^*(g)$ and $\mathscr{K}_s^*(g)$ of analytic functions. we obtain coefficient bounds for $f(z)$ when $f(z)$ is in the class $\mathscr{S}_g^*$ or is in the class $\mathscr{K}_g^*$. These results generalize many known results.

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1. Introduction

Let $\mathbb{C}$ be the set of complex numbers ,and

$$\mathbb{N} = \{1, 2, 3, \cdots\}$$

be the set of positive integers. We also let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

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We denote by $\mathscr{S}$ the subclass of the analytic function class $\mathscr{A}$ consisting of all functions in $\mathscr{A}$ which are also univalent in $\mathbb{U}$. For two functions f and g, analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$(written $f \prec g$)if there exists a Schwarz function $\mathfrak{w}(z)$, analytic in $\mathbb{U}$ with

$$\mathfrak{w}(0) = 0 \text{ and } |\mathfrak{w}(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\mathfrak{w}(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In many earlier investigations various interesting subclasses of the analytic function class $\mathscr{A}$ and the univalent function class $\mathscr{S}$ have been studied from a number of different viewpoints. We choose to recall here the investigations by (for example) Srivastava *et al* ([1], [2] and [3]), Breaz *et al*.[4], Owa *et al*. [5], In particular, Sakaguchi [6] introduced a subclass $\mathscr{S}_s^*$ of analytic functions.

Definition 1. *([6]). A function $f(z) \in \mathscr{A}$ is said to belong to the class $\mathscr{S}_s^*$ of starlike with respect to symmetric points in $\mathbb{U}$ if it satisfies the following inequality:*

$$\Re\left\{\frac{2zf'(z)}{f(z) - f(-z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

Then, Goel and Mehrok in 1982 introduced a subclass of $\mathscr{S}_s^*$ which were denoted by $\mathscr{S}_s^*(A,B)$.

Definition 2. *(see [7]) A function $f(z) \in \mathscr{A}$ is said to belong to the class $\mathscr{S}_s^*(A,B)$ if it satisfies the following condition:*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U};\ -1 \le B < A \le 1).$$

Recently, Aini Janteng and Suzeini Abdul [8] extended Definition 2 by introducing the following subclass of analytic functions.

Definition 3. *(see [8]) Let the function $f(z)$ be analytic in $\mathbb{U}$ and defined by (1). We say that $f \in \mathscr{K}_s^*(A,B)$ if there exists a function $h(z) \in \mathscr{S}_s^*(A,B)$ such that*

$$\frac{2zf'(z)}{h(z) - h(-z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U};\ -1 \le B < A \le 1).$$

Here, in our present sequel to some of the aforecited works (especially [7] and [8]), we introduce the following subclass of analytic functions.

Definition 4. *Let $g : \mathbb{U} \to \mathbb{C}$ be a convex function such that $g(0) = 1$, $g(\bar{z}) = \overline{g(z)}$, for $z \in \mathbb{U}$, $\Re(g(z)) > 0$ on $z \in \mathbb{U}$. Let f be an analytic function in $\mathbb{U}$ defined by (1). We say that $f \in \mathscr{S}_s^*(g)$, if it satisfies the following condition:*

$$\frac{2zf'(z)}{f(z) - f(-z)} \in g(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Definition 5. *Let g satisfy the conditions of Definition 4 and f be an analytic function in $\mathbb{U}$ defined by (1). We say that $f \in \mathscr{K}_s^*(g)$ if there exists a function $h(z) \in \mathscr{S}_s^*(g)$ such that*

$$\frac{2zf'(z)}{h(z) - h(-z)} \in g(\mathbb{U}) \quad (z \in \mathbb{U}).$$

Remark 1. *There are many choices of the function g which would provide interesting subclasses of analytic functions. For example, if we let*

$$g(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U};\ -1 \leq B < A \leq 1),$$

then it is easy to verify that g satisfies the hypotheses of Definition 4. So, by taking

$$g(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U};\ -1 \leq B < A \leq 1)$$

in Definitions 4 and 5, we easily observe that the function classes

$$\mathscr{S}_s^*(g) \text{ and } \mathscr{K}_s^*(g)$$

become the aforementioned function classes

$$\mathscr{S}_s^*(A, B) \text{ and } \mathscr{K}_s^*(A, B),$$

respectively.

In this paper, by using the principle of subordination, we obtain coefficient bounds for functions in the subclasses $\mathscr{S}_s^*(g)$ and $\mathscr{K}_s^*(g)$. Our results would unify and extend the corresponding works of some authors.

2. Main Results and Their Proofs

In order to prove our main results, we first recall the following lemma due to Rogosinski.

Lemma 1. *Let the function g given by*

$$g(z) = \sum_{k=1}^{\infty} g_k z^k \quad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. *Suppose also that the function* $f(z)$ *given by*

$$f(z)=\sum_{k=1}^{\infty}a_k z^k \quad (z\in\mathbb{U})$$

be holomorphic in $\mathbb{U}$. *If* $f(z)\prec g(z)$ $(z\in\mathbb{U})$*, then*

$$|a_k|\leq|g_1| \quad (k\in\mathbb{N}).$$

We now state and prove the main results of our present investigation.

Theorem 1. *Let the function* $f(z)\in\mathscr{A}$ *be given by (1). If* $f\in\mathscr{S}_s^*(g)$*, then*

$$|a_{2n+1}|\leq\frac{|g'(0)|}{n!2^n}\prod_{j=1}^{n-1}(|g'(0)|+2j) \quad (n\in\mathbb{N}), \tag{2}$$

and

$$|a_{2n}|\leq\frac{|g'(0)|}{n!2^n}\prod_{j=1}^{n-1}(|g'(0)|+2j) \quad (n\in\mathbb{N}). \tag{3}$$

Proof. First we prove (2) using the principle of mathematical induction.

Let

$$p(z)=\frac{2zf'(z)}{f(z)-f(-z)}. \tag{4}$$

Since $f\in\mathscr{S}_s^*(g)$, it follows that

$$p(0)=g(0)=1 \text{ and } p(z)\in g(\mathbb{U}) \quad (z\in\mathbb{U}).$$

Therefore, we have

$$p(z)\prec g(z)\ (z\in\mathbb{U}),$$

where

$$p(z)=1+p_1z+p_2z^2+\ldots$$

According to Lemma 1, we obtain

$$|p_i|\leq|g'(0)|\ (i\in\mathbb{N}). \tag{5}$$

From (4), we deduce that

$$\begin{aligned}&z+2a_2z^2+3a_3z^3+\ldots+2na_{2n}z^{2n}+(2n+1)a_{2n+1}z^{2n+1}+\ldots\\&=[z+a_3z^3+a_5z^5+\ldots+a_{2n-1}z^{2n-1}+a_{2n+1}z^{2n+1}+\ldots](1+p_1z+p_2z^2+\ldots).\end{aligned}$$

Equating the coefficients of the same powers of z , we obtain that

$$2na_{2n+1}=p_{2n}+p_{2n-2}a_3+\ldots+p_2a_{2n-1} \quad (n\in\mathbb{N}^*:=\mathbb{N}\setminus\{1\}=\{2,3,4,\ldots\}) \tag{6}$$

and

$$2na_{2n} = p_{2n-1} + p_{2n-3}a_3 + \ldots + p_1 a_{2n-1} \quad (n \in \mathbb{N}^*). \tag{7}$$

Combining(3), (4) and (5), for $n = 1, 2$, we obtain

$$|a_2| \le \frac{|g'(0)|}{2}, |a_3| \le \frac{|g'(0)|}{2}, |a_4| \le \frac{|g'(0)| \cdot (|g'(0)| + 2)}{2 \times 4} \tag{8}$$

and

$$|a_5| \le \frac{|g'(0)| \cdot (|g'(0)| + 2)}{2 \times 4}, \tag{9}$$

respectively.

According to Lemma 1 and (5), we obtain that

$$|a_{2n+1}| \le \frac{|g'(0)|}{2n} \left[1 + \sum_{k=1}^{n-1} |a_{2k+1}| \right] \quad (n \in \mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, 4, \ldots\}). \tag{10}$$

We assume that (2) holds for $k = 3, 4, \ldots (n-1)$. Then from (8), we obtain

$$|a_{2n+1}| \le \frac{|g'(0)|}{2n} \left[1 + \sum_{k=1}^{n-1} \frac{|g'(0)|}{k!2^k} \prod_{j=1}^{k-1} (|g'(0)| + 2j) \right].$$

To this end, it is sufficient to show that

$$\frac{|g'(0)|}{2m} \left[1 + \sum_{k=1}^{m-1} \frac{|g'(0)|}{k!2^k} \prod_{j=1}^{k-1} (|g'(0)| + 2j) \right] = \frac{|g'(0)|}{m!2^m} \prod_{j=1}^{m-1} (|g'(0)| + 2j) \, (m = 3, 4, \ldots, n). \tag{11}$$

It is elementary to verify that (2) is valid for $m = 3$.

Let us suppose that (2) is true for all m, $3 < m \le (n-1)$. Then form (9)

$$\begin{aligned}
&\frac{|g'(0)|}{2n} \left[1 + \sum_{k=1}^{n-1} \frac{|g'(0)|}{k!2^k} \prod_{j=1}^{k-1} (|g'(0)| + 2j) \right] \\
&= \frac{2(n-1)}{2n} \frac{|g'(0)|}{2(n-1)} \left[1 + \sum_{k=1}^{n-2} \frac{|g'(0)|}{k!2^k} \prod_{j=1}^{k-1} (|g'(0)| + 2j) \right] + \frac{|g'(0)|}{2n} \frac{|g'(0)|}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (|g'(0)| + 2j) \\
&= \frac{2(n-1)}{2n} \frac{|g'(0)|}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (|g'(0)| + 2j) + \frac{|g'(0)|}{2n} \frac{|g'(0)|}{(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (|g'(0)| + 2j) \\
&= \frac{|g'(0)|}{(2n)(n-1)!2^{n-1}} \prod_{j=1}^{n-2} (|g'(0)| + 2j)(|g'(0)| + 2(n-1))
\end{aligned}$$

$$= \frac{|g'(0)|}{n!2^n} \prod_{j=1}^{n-1} (|g'(0)| + 2j).$$

Thus, (9) holds for $m = n$ and hence (2) follows. with the similar method and reasoning as in the proof of (2) , we also prove that (3) holds. This completes the proof of Theorem 1 .

Theorem 2. *Let the function $f(z) \in \mathscr{A}$ be given by (1). If $f \in \mathscr{K}_s^*(g)$, then*

$$|a_{2n}| \leq \frac{|g'(0)|}{n!2^n} \prod_{j=1}^{n-1} (|g'(0)| + 2j) \quad (n \in \mathbb{N})$$

and

$$|a_{2n+1}| \leq \frac{|g'(0)|}{n!2^n} \prod_{j=1}^{n-1} (|g'(0)| + 2j) \quad (n \in \mathbb{N}).$$

Proof. Theorem 2 can be proven by using similar arguments as in the proof of Theorem 1, so we choose to omit the details involved.

3. Corollaries and Consequences

In view of Remark 1, if we set

$$g(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}; \quad -1 \leq B < A \leq 1)$$

in Theorems 1 and 2, we obtain easily to Corollaries 1 and 2, respectively.

Corollary 1. *Let the function $f(z) \in \mathscr{A}$ be given by (1). If $f \in \mathscr{S}_s^*(A, B)$, then*

$$|a_{2n}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j) \quad (n \in \mathbb{N})$$

and

$$|a_{2n+1}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j) \quad (n \in \mathbb{N}).$$

Corollary 2. *Let the function $f(z) \in \mathscr{A}$ be given by (1). If $f \in \mathscr{K}_s^*(A, B)$, then*

$$|a_{2n}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j) \quad (n \in \mathbb{N})$$

and

$$|a_{2n+1}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j) \quad (n \in \mathbb{N}).$$

Remark 2. *Corollaries 1 and 2 were proven earlier by Goel and Mehrok [7] and Aini Janteng and Suzeini Abdul [8], respectively. However, we are able to derive these results easily as consequences of Theorems 1 and 2.*

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