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Burst Error Enumeration of m-Array Codes over Rings and Its Applications

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Abstract. Enumerating burst errors enables to obtain bounds on parameters of codes. Recently, Jain in [5] established a Reiger's type bound for burst error correcting matrix codes over finite fields with respect to a non Hamming metric. Here, we extend these results to array codes over finite rings. Further, we also introduce a new constructive method for counting burst errors that avoids solving Diophantine inequalities in order to compute burst errors for each given weight. Finally, we apply our results on establishing some bounds for array codes over finite rings.

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1. Introduction

The RT (non Hamming) metric for m-array codes has gained quite interest recently. On the other hand, burst error correction of array codes is also another important topic investigated by several researchers lately [1, 11, 2]. In [5], the author emphasizes the importance of considering burst errors by giving an application of array codes with respect to the RT metric. By enumerating burst errors of particular weights, a Rigger's Type bound is established. Recently, an alternative approach that relies on generating type of multivariable polynomials in order of computing the number of a class of burst errors is presented in [10]. Here, we generalize these results to array codes over finite rings. In [5], counting of burst errors over fields relies on solving Diophantine inequalities 2. Further, this computation needs to be carried out for each particular weight when the question is to compute the number of burst errors of a particular weight or less which is the case. Here, we also introduce a new constructive method for counting burst errors that avoids solving Diophantine inequalities in order to compute burst errors for each given weight. Moreover, we introduce a multivariable polynomial whose coefficients enumerate the number of burst errors of particular weight and hence this

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method avoids recalculations. Finally, we apply our results on establishing some bounds for array codes over finite rings.

The *RT* (non Hamming) metric for array (matrix) codes over fields is defined in [7] and some bounds for the minimum distance are established. Some applications of this metric to uniform distributions are given in [8]. A MacWilliams type identity for codes over matrices with respect to the RT metric is proven in [3]. Further, a MacWilliams type identity for complete weight enumerators of codes over matrices with respect to the RT metric is proven in [9].

R be a commutative finite ring with unity. Throughout the paper we assume that the cardinality of R equals to q.

Definition 1. *Let $M = M_{m\times s}(R)$ be the set of $m \times s$ matrices with components from R . A subset C of M is called an m-array code. If C is a linear R-submodule, then C is called a linear m-array code.*

In this paper we will always refer to linear array codes.

Definition 2 (Non Hamming-RT weight)**.** *Let $x = (x_1, x_2, \ldots, x_s) \in R^s$. The RT weight (or ρ-weight) of x is defined by*

$$w_N(x) = \begin{cases} max\{i|x_i \neq 0\}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Let $A \in M_{m\times s}(R)$ and A_i be the ith row of the matrix A. Then the RT weight of the matrix A is the sum of the RT weights of its rows, in other words $w_N(A) = \sum_{i=1}^{m} w_N(A_i)$.

The RT metric (ρ distance) is defined by $\rho(x, y) = w_N(x - y)$ where $x, y \in M$.

Note the difference between the weight and distance notations of Hamming and RT (Non Hamming) metrics. The letter "N" for RT metric is used to emphasize the non Hamming case.

Definition 3. *Let C be an R-linear code. The minimum nonzero ρ distance between the codewords of C is denoted by $d_N(C)$. The minimum nonzero ρ weight among all codewords of C is denoted by $w_N(C)$. In linear case, $d_N(C) = w_N(C)$, and $d_N(C)$ is called the minimum distance of C with respect to the RT metric.*

The concept of burst errors for codes in a classical setup is introduced in [4]. Recently, in [5], the notion of burst errors for matrix codes with respect to RT metric has been introduced and some formulas on enumeration of burst errors has been obtained and by use of this enumeration Reiger's type bounds are stated and proved. We extend the definitions introduced by Jain in [5] from finite field case to finite commutative ring case. Work on linear codes over rings is an interesting and ongoing topic in coding theory. Mostly, the work is done over finite rings such as $\mathbb{Z}_m$, Galois rings, chain rings and etc. In this paper, all this well known rings are covered.

In the introduction, the basics and definitions are covered. In Section 2, generic burst errors are defined. An equivalence relation on the set of burst errors is introduced. The equivalence classes are shown to be the generic burst errors. Later, a generic multi variable polynomial that represents all generic burst errors is introduced. By substituting suitable variables in a generic multivariable polynomial, we show how to obtain a new polynomial that gives the number of burst errors and their weights. Finally, applying a final substitution, we obtain a new polynomial called the burst weight enumerator polynomial with coefficients being the number of burst errors that correspond to the powers which gives the RT weights. Thus, the new computation method of burst errors of order $p \times r$ in the space $M_{p\times r}(R)$ (the set of matrices of order $p \times r$ with entries from a finite commutative ring with q elements R) is given. In Section 3, the computation method of burst errors of order $p \times r$ in the space $M_{m\times s}(R)$ where $1 \le p \le m, 1 \le r \le s$ is presented by making use of the results obtained in Section 2. In Section 4, some applications for obtaining bounds are presented. Finally, the paper is concluded by some remarks.

Definition 4. *A burst of order pr or $(p \times r)$ $(1 \le p \le m, 1 \le r \le s)$ in the space $M_{m\times s}(R)$ is an $m\times s$ matrix in which all the nonzero entries are confined to some $p\times r$ submatrix which has non zero first and last rows as well as nonzero first and last columns. $B^{p\times r}_{m\times s}(R)$ denotes the number of burst errors of order $p \times r$.*

The number $B^{p\times r}_{m\times s}(F_q)$ that is used in establishing some bounds for array codes over a finite field F_q is presented by Jain in [5] with the following theorem:

Theorem 1 ([5])**.** *Let $B^{p\times r}_{m\times s}(F_q)$ denote the number of bursts of order pr in $M_{m\times s}(F_q)$. Then,*

$$B^{p\times r}_{m\times s}(F_q) = \begin{cases} ms(q-1), & p=1, r=1, \\ m(s-r+1)(q-1)^2 q^{r-2}, & p=1, r\ge 2 \\ (m-p+1)s(q-1)^2 q^{p-2}, & p \ge 2, r=1, \\ (m-p+1)(s-r+1)q^{r(p-2)}[(q^r-1)^2 & \\ \quad -2(q^{r-1}-1)^2 q^{2-p} + (q^{r-2}-1)^2 q^{4-2p}], & p\ge 2, r \ge 2. \end{cases}$$

Further, in [5] a formula for the number of bursts of a particular order and not exceeding a given ρ-weight is stated and proved in the following theorem:

Theorem 2 ([5])**.** *The number of bursts of order pr $(1 \le p \le m, 1 \le r \le s)$ in $M_{m\times s}(F_q)$ having ρ-weight w or less $(1 \le w \le ms)$ is given by*

$$B^{p\times r}_{m\times s}(F_q, w) = \begin{cases} m(q-1)\min(w,s), & p=1, r=1, \\ m\min(w-r+1, s-r+1)(q-1)^2 q^{r-2}, & p=1, r\ge 2, \\ (m-p+1)B_3, & p\ge 2, r=1, \\ (m-p+1)B_4, & p \ge 2, r\ge 2. \end{cases}$$

where

$$B_3 = \sum_{j=1}^{\min([w/2],s)} \sum_{\eta=0:\eta j\le w-2j}^{p-1} (q-1)^2 \binom{p-2}{\eta} (q-1)^{\eta},$$

$$B_4 = (m-p+1) \sum_{j=1}^{\min(w-r+1, s-r+1)} (L_j^p - 2L_j^{p-1} + L_j^{p-2}),$$

and

$$L_j^p = \sum_{k_j,\dots,k_{j+r-1}} \frac{p!}{\prod_{l=0}^{r-1} k_{j+1}! \left(p - \sum_{l=0}^{r-1} k_{j+l}\right)!} \left(\frac{q-1}{q}\right)^{\sum_{l=0}^{r-1} k_{j+l}} q^{\sum_{l=0}^{r-1}(l+1)k_{j+l}} \tag{1}$$

where $k_j, k_{j+1}, \dots, k_{j+r-1}$ *being nonnegative integers such that*

$$k_j > 0, \quad k_{j+1}, k_{j+2}, \dots, k_{j+r-2} \geq 0, k_{j+r-1} > 0,$$
$$\sum_{l=0}^{r-1} k_{j+l} \leq p$$
$$\sum_{l=0}^{r-1} (j+l)k_{j+l} \leq w. \tag{2}$$

In Theorem 2, computing the number of burst errors of a particular order is still a challenging task. In Equation 2, the two Diophantine inequalities are first to be solved in the set of natural numbers. Then, the L_j^p numbers are computed by using the k_i solutions. In [5], some examples using this approach are worked out explicitly. In the next sections, we first extend the results obtained by Jain in [5] to array codes over R by introducing a new constructive method that gives the number of burst errors. This new method, it does not only give the number of a particular burst error weight but it also gives all spectra of the weights in a single computation. The spectra of the number of burst errors shall be called the burst error weight enumerator. Finally, we apply our results to obtain new bounds for array codes over rings.

2. Enumerating Burst Errors

In this section, we shall work on the space $M_{p\times r}(R)$ and consider only burst errors of order $p \times r$. In the next section, we shall consider burst errors of order $p \times r$ in the space $M_{m\times s}(R)$ where $1 \leq p \leq m, 1 \leq r \leq s$.

In order to introduce the new approach for computing burst error matrices we need to state some definitions and introduce some new concepts.

Definition 5. *If* $A \in M_{m\times s}(F_q)$ *and* $w_N(A_i) = \alpha_i$, *then the matrix* A *is said to have a weight distribution of type* $(\alpha_1, \alpha_2, \dots, \alpha_m)$.

If A_j is the jth row of an $A \in M_{p\times r}$ matrix, then the index of A_j denoted by (k_j, l_j) where $1 \leq k_j \leq l_j \leq r$ is defined by $a_{ji} = 0$ for all $1 \leq i \leq k_j - 1$ and $a_{jk_j} \neq 0$ and $l_j = w_N(A_j)$. The index of a matrix with rows $A_1, A_2, \dots A_p$ is defined by $((k_1, l_1), (k_2, l_2), \dots, (k_p, l_p))$. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ a_{21} & \cdots & a_{2r} \\ \vdots & \vdots & \vdots \\ a_{p1} & \cdots & a_{pr} \end{pmatrix}.$$

The first row and column, and the last row and column of the matrix A are shown in the following rectangle which is called the frame of matrix A.

$$\begin{array}{|c|ccc|c|}
\hline
a_{11} & a_{12} & \cdots & a_{1r-1} & a_{1r} \\
\hline
a_{21} & \cdots & \cdots & \cdots & a_{2r} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
a_{p1} & a_{p2} & \cdots & a_{pr-1} & a_{pr} \\
\hline
\end{array}$$

The restrictions on the first and last rows together with the first and last columns determine whether a matrix is a burst error or not. Hence, we focus on the frame of the matrix and introduce some new definitions in order to control these entries and transform the problem of computing the number of these errors into an algebraic problem. Further, the four entries a_{11}, a_{1r}, a_{p1} and a_{pr} of A shall be referred as the corners of the matrix.

Definition 6. *A generic burst error $A \in M_{p\times r}$ of order $p \times r$ is a burst error of order $p \times r$ such that the first and the last rows have exactly two or one nonzero entries, and the submatrix of size $p-2 \times r-2$ that is obtained by removing first and last rows and first and last columns (i.e. the borders) has all entries equal to zeroes, i.e, the entries that fall out off the frame are all equal to zeroes.*

Example 1.

$$B_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \quad B_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{4\times 3} \quad B_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}_{2\times 3}.$$

The matrices B_1, B_2 and B_3 are generic burst errors. Note that the matrices $E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{2\times 3}$ and $D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$ are burst errors but they are not generic burst errors. However, the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}_{3\times 3}$ is a generic burst error.

In order to determine the generic burst errors, we need to classify matrices according to their corners.

Let $A = (a_{ij}) \in M_{p\times r}(R)$ be a generic burst error and the jth row $A_j = (a_{j1}, a_{j2}, \ldots, a_{jr})$. By definition, the jth row has at most two nonzero entries. Let (k_j, l_j) be the index of A_j. We associate a multi variable term to the jth row of A in the following way:

$$\mu(A_j) = \begin{cases} z_j, & 1 = k_j = l_j \\ x_j^{k_j}, & 1 < k_j = l_j < r \\ y_j, & 1 < k_j = l_j = r \\ x_j^{k_j} X_j^{l_j}, & k_j < l_j < r \\ x_j^{k_j} y_j, & 1 < k_j < l_j = r \\ z_j X_j^{l_j}, & 1 = k_j < l_j < r \\ z_j y_j, & 1 = k_j < l_j = r. \end{cases}$$

We associate z_j and y_j variables for the first and last column entries respectively. If there exist two nonzero entries in the first or last row which are different from the corner entries, then we associate $x_j^k X_j^l$ where the small letter indicates the beginning and capital X_j indicates the end of the nonzero entries. We use x_j^i, when the jth row has only one nonzero entry on the i-th entry, $1 < i < r$.

In a natural way, we extend this representation to the matrix A by taking the product of all terms $\mu(A_j)$ corresponding to the rows of A. The terms that correspond to the rows different from the first and last contain only the terms composed by z and y variables.

For example, the representations of the following matrices are given below:

Example 2.

$$\begin{array}{cccc} \textit{Generic errors} & A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}. \\ \textit{The terms} & x_1^2 z_2 y_3 & z_1 y_1 x_3^2 & x_1^2 X_1^4 z_2 y_2 \end{array}$$

Given a p multi variable representation of a generic burst error, it is possible to list all burst errors by using it. In general, we have

$$G = \begin{array}{c|c|c} a_{11} & \cdots & a_{1r} \\ \hline \vdots & \vdots & \vdots \\ \hline a_{p1} & \cdots & a_{pr} \end{array} \quad \longleftrightarrow \quad \begin{array}{c|c|c} z_1 & \cdots x_1^j, x_{k_1} X^{l_1}, X^{l_1} \cdots & y_1 \\ \hline \vdots z_i \vdots & \vdots & \vdots y_i \vdots \\ \hline z_p & \cdots x_p^j, x_{k_p} X^{l_p}, X^{l_p} \cdots & y_p \end{array}$$

and the term $\prod_{j=1}^{p} \mu(A_j)$ corresponds to the term of the matrix A.

We classify generic burst errors by considering their corners. There are $2^4 = 16$ possible cases for these corners and corresponding multi variable terms. We list them in Table 1.

In order to solve the problem of representing burst array errors in terms, we need to split it into cases. First, we need to split it to two main cases as $p \geq 3$ and $r \geq 3$ and otherwise. We work out these cases by the following theorems.

Let $\tilde{z} = (z_1, \ldots, z_p)$, $\tilde{x} = (x_1, \ldots, x_p)$, $\tilde{y} = (y_1, \ldots, y_p)$ and $\tilde{X} = (X_1, \ldots, X_p)$.

Definition 7. *Let $K_{p\times r}$ be the set of all generic burst errors of size $p \times r$. Let $G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = \sum_{A\in K} \prod_{j=1}^{p} \mu(A_j)$ be the multivariable polynomial whose terms represent generic burst errors.*

$(a_{11}, a_{1r}, a_{p1}, a_{pl})$	term		$(a_{11}, a_{1r}, a_{p1}, a_{pl})$	term
$(0,0,0,0)$	1		$(0,1,1,0)$	$y_1 z_p$
$(1,0,0,0)$	z_1		$(0,1,0,1)$	$y_1 y_p$
$(0,1,0,0)$	y_1		$(0,0,1,1)$	$z_p y_p$
$(0,0,1,0)$	z_p		$(1,1,1,0)$	$z_1 y_1 z_p$
$(0,0,0,1)$	y_p		$(1,1,0,1)$	$z_1 y_1 y_p$
$(1,1,0,0)$	$z_1 y_1$		$(1,0,1,1)$	$z_1 z_p y_p$
$(1,0,1,0)$	$z_1 z_p$		$(0,1,1,1)$	$y_1 z_p y_p$
$(1,0,0,1)$	$z_1 y_p$		$(1,1,1,1)$	$z_1 y_1 z_p y_p$

Table 1: Terms with respect to the corners

In the following theorem we first determine multivariable polynomial G that gives the terms of generic burst errors of size $p \leq 2$ and $r \leq 2$. The remaining case is treated separately in the next theorem.

Theorem 3. *1. Let $p = 1$ and $r \geq 2$. Then,*

$$G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = z_1 y_1.$$

2. Let $p \geq 2$ and $r = 1$. Then,

$$G(\tilde{z}) = z_1 z_p.$$

3. Let $p = 2$ and $r \geq 2$. Let $S = \{(i,j,k,l) \in Z_2^4 | (i,k) \neq (0,0), (j,l) \neq (0,0)\}$. Then,

$$G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = \sum_{(i,j,k,l)\in S} z_1^i y_1^j z_2^k y_2^l \gamma_{ij}(X_1)\gamma_{kl}(X_p).$$

4. Let $p \geq 2$ and $r = 2$.
Let $T = \{(i,j,k,l) \in \{0,1\}^4 | (i,j) \neq (0,0), (k,l) \neq (0,0)\}$. Then,

$$G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = \sum_{(i,j,k,l)\in T} z_1^i y_1^j z_p^k y_p^l \overline{Z}_{ik} \overline{Y}_{jl}.$$

where

$$\overline{Z}_{st} = \begin{cases} -1 + \prod_{i=2}^{p-1}(1 + z_i), & (s,t) = (0,0) \\ \prod_{i=2}^{p-1}(1 + z_i), & \textit{otherwise} \end{cases}$$

$$\overline{Y}_{st} = \begin{cases} -1 + \prod_{i=2}^{p-1}(1 + y_i), & (s,t) = (0,0) \\ \prod_{i=2}^{p-1}(1 + y_i), & \textit{otherwise} \end{cases}$$

$$\gamma_{st}(X_k) = \begin{cases} \sum_{i=2}^{r-2} x_k^i \left(1 + \sum_{j=i+1}^{r-1} X_k^j\right) + x_k^{r-1}, & (s,t) = (0,0) \\ 1 + \sum_{i=2}^{r-1} x_k^i, & (s,t) = (0,1) \\ 1 + \sum_{i=2}^{r-1} X_k^i, & (s,t) = (1,0) \\ 1, & (s,t) = (1,1). \end{cases}$$

Proof. The cases 1 and 2 follow directly from the definitions. We give the proof of case 3 and the proof of the case 4 also can be shown by using similar arguments. Let $p=2$ and $r\geq 2$ and $A=(a_{ij})\in M_{2\times r}(R)$ be a generic burst error. Hence the matrix A has only two rows and $r\geq 2$ columns. There are $2^4=16$ possible values (as zero and none zero) for the corners of A. Since the matrix A is a burst error, the first and last column must be nonzero. The first column is equal to zero if $(i,k)=(0,0)$ and the last column is equal to zero if $(j,l)=(0,0)$. Excluding these seven cases the possible values for the corners are all values of the set S.

If both corners of the first row are equal to zero, then the term that corresponds to A must consist of $\beta = x_1^2(X_1^2+\cdots+X_1^{r-1})+x_1^3(X_1^4+\cdots+X_1^{r-1})+\cdots+x^{r-1}$. Otherwise, the term that corresponds to A must consist of $1+\beta$. These two cases are represented by the multiple $\gamma_{ij}(X_1)$. In a similar way we can argue for the last row.

Example 3. *The following generic multivariable polynomial G gives the term representation of $A\in M_{2\times 3}(F_2)$ generic burst errors of order 2×3. By Theorem 3 part 3, we have*

$$\begin{aligned}
G(\tilde{z},\tilde{x},\tilde{X},\tilde{y}) &= \sum_{(i,j,k,l)\in S} z_1^i y_1^j z_2^k y_2^l \gamma_{ij}(X_1)\gamma_{kl}(X_2) = z_1y_1X_2^2+z_2y_2X_1^2\\
&\quad + z_1y_2(1+X_1^2)(1+x_2^2)+z_2y_1(1+x_1^2)(1+X_2^2)+z_1y_1y_2(1+x_2^2)\\
&\quad + y_1z_2y_2(1+X_1^2)+z_1z_2y_1(1+X_2^2)+z_1z_2y_2(1+X_1^2)+z_1y_1z_2y_2\\
&= z_1y_1X_2^2+z_2y_2x_1^2+z_1y_2+z_1y_2x_2^2+z_1y_2X_1^2+z_1y_2X_1^2x_2^2\\
&\quad + z_2y_1+z_2y_1x_2^2+z_2y_1X_1^2+z_2y_1X_1^2x_2^2+z_1y_1y_2+z_1y_1y_2X_2^2\\
&\quad + y_1z_2y_2+y_1z_2y_2x_1^2+z_1z_2y_1+z_1z_2y_1X_2^2+z_1z_2y_2+z_1z_2y_2X_1^2\\
&\quad + z_1y_1z_2y_2.
\end{aligned}$$

Using the generic multivariable polynomial, we can list all generic burst errors of size 2×3.

$$\begin{pmatrix}1&0&1\\0&1&0\end{pmatrix},\quad \begin{pmatrix}0&1&0\\1&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&0\\0&1&1\end{pmatrix},\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix},$$
$$\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix},\quad \begin{pmatrix}0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&0\end{pmatrix},\quad \begin{pmatrix}0&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&1\\0&1&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&1\end{pmatrix},\quad \begin{pmatrix}0&1&1\\1&0&1\end{pmatrix},\begin{pmatrix}1&0&1\\1&0&0\end{pmatrix},$$
$$\begin{pmatrix}1&0&1\\1&1&0\end{pmatrix},\quad \begin{pmatrix}1&0&0\\1&0&1\end{pmatrix},\begin{pmatrix}1&1&0\\1&0&1\end{pmatrix},\quad \begin{pmatrix}1&0&1\\1&0&1\end{pmatrix}.$$

Theorem 4. *All generic bursts of order $p\times r$ $(p,r\geq 3)$ are obtained as terms of the following multi variable polynomial, say generic multivariable polynomial:*

$$G(\tilde{x},\tilde{z},\tilde{X},\tilde{y}) = \sum_{(i,j,k,l)\in Z_2^4} z_1^i y_1^j z_p^k y_p^l \overline{Z}_{ik}\overline{Y}_{jl}\gamma_{ij}(X_1)\gamma_{kl}(X_p).$$

Proof. Let $A = (a_{ij}) \in M_{p\times r}(R)$ be a generic burst error.

$$A = \begin{array}{c|c|c} a_{11} & \cdots & a_{1r} \\ \hline \vdots & \text{zero submatrix} & \vdots \\ \hline a_{p1} & \cdots & a_{pr} \end{array}.$$

We consider the corners of A. There are 16 cases depending on the corner's values as whether they are equal to one or zero. If any pair of adjacent corners corresponding to the first row, column or last row or column both equal to zero, then at least one of the entries between the corners must equal to one. As argued in the proof of Theorem 3, $\gamma_{ij}(X_1)$ represents the term that corresponds to the first row. Similarly, $\gamma_{kl}(X_p)$ represents the term that corresponds to the last row. Again, if the first and the last entry of the first column both equal to zero, then the term corresponding to the matrix has to include the factor $-1+\prod_{i=2}^{p-1}(1+z_i)$ since at least one of the entries must equal to one excluding the zero column. In other cases, $\prod_{i=2}^{p-1}(1+z_i)$ represents the first column. Arguing similarly for the last column, we get the result.

Example 4. *The following generic multivariable polynomial G gives the term representation of $A \in M_{3\times 3}(F_2)$ generic burst errors of order* 3×3 :

$$G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = \sum_{(i,j,k,l)\in\{0,1\}^4} z_1^i y_1^j x_3^k y_3^l \overline{Y}_{jl}(\alpha)\gamma_{ij}(X_1)\gamma_{kl}(X_3).$$

Now, we shall make use of generic burst errors to determine all burst errors especially the number of burst errors.

First, we introduce a relation on burst errors that is based on the frames of matrices.

Definition 8. *Let $\mathbb{B}$ be the set of all burst errors of order $p \times r$. Let $A, B \in \mathbb{B} \subset M_{p\times r}(R)$. It is said that the matrix A is related to the matrix B, i.e $A \approx B$, in the set $\mathbb{B}$ if and only if $\mu(A_j) = \mu(B_j)$ for $j = 1, p$ and $\mu(A_j)|_{x_j=X_j=1} = \mu(B_j)|_{x_j=X_j=1}$ for $j \neq 1, p$.*

The proof of the following Lemma follows from the definitions.

Lemma 1. *The relation "$\approx$" defined on the set $\mathbb{B}$ is an equivalence relation.*

Now we have a partition of the space of burst errors into disjoint classes which are generic burst errors. Let C_A represent the set of equivalence class of matrix A. We determine the representative set C_A in the following way: Let A_1 be the first row of a generic burst error matrix A. If the weight of the row is two (the term consists of z_1y_1, $z_1X_1^{l_1}$, $x_1^{k_1}X_1^{l_1}$ or $x_1^{k_1}y_1$), then these two entries are nonzero and the choices for the entries in between run through all nonzero elements of the ring R. If the weight is equal to one (the term consists of y_1, z_1 or $x_1^{k_1}$ only), then a nonzero choice for this entry from $R\backslash\{0\}$. In a similar way we can argue for the last row of the matrix. Also we note that the entries that fall out off the frame can take any value without any restriction.

Now we use generic burst error matrices in Example 3, to find all burst errors of size 2×3 over F_2. We list the equivalency classes with respect to the relation:

$$\left\{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\right\},$$

$$\left\{\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\right\}.$$

The sets are classes that correspond to a generic burst error. Altogether, they add up to 32 burst errors of order 2×3 over F_2. If we were working over the field F_3, then, for example, the generic burst error class that correspond to the generic matrix $Z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ would be;

$$C_Z = \left\{\begin{pmatrix} a & b & c \\ 0 & d & 0 \end{pmatrix} | a, c, d \in F_3 \backslash \{0\} \text{ and } b \in F_3\right\}$$

and $|C_Z| = 2^3 \cdot 3 = 24$. In general, over a ring R, the generic matrix $Z \in M_{2\times 3}(R)$ represented by the term $x_1 y_1 X_2^2$ will have a burst error class of size $(q-1)^3 q$.

Corollary 1. *The number of terms of generic polynomial G, gives the number of equivalence classes of equivalence relation "≈" which are generic burst errors.*

Under this observation, we can make use of generic burst terms and hence generic multi variable polynomial to obtain the number of burst errors.

Definition 9. *Let $A \in \mathbb{B} \subset M_{p\times r}(R)$. Assume that A has a weight distribution of type $(\alpha_1, \alpha_2, \ldots, \alpha_p)$. We associate a term $X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_p^{\alpha_p}$ to the matrix A. Then, the p-variable polynomial*

$$H_{p\times r}(\tilde{X}) = \sum_{A\in\mathbb{B}} X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_p^{\alpha_p}$$

is said to be the weight spectra polynomial of the burst errors in $M_{p\times r}(R)$.

Now we shall make use of generic polynomial in order to compute the weight spectra polynomial of burst errors. Here, we point out that the frame of the arrays hence generic bursts lead to construction of burst errors. So, if we have for instance a term $z_1^{a_1} x_1^{b_1} X_1^{c_1} y_1^{d_1}$ for the first row, then the first and the last if any nonzero entry will lead to $(q-1)^2$ choices and there is no restriction in between with q choices for each entry. However, if this row consists of a single nonzero entry then the total number of choices is equal to $q-1$. In order to accommodate all cases we will invent the following compact substitution. Assume that the term $z_1^{a_1} x_1^{b_1} X_1^{c_1} y_1^{d_1}$ is related to the first row of a generic burst error with the convention that the powers may equal to zero in generic polynomial. The term $z_1^{a_1} x_1^{b_1} X_1^{c_1} y_1^{d_1}$ that corresponds to the first row of the generic matrix leads to

$$(q-1)^{2\left[w(a_1d_1)+w(a_1c_1)+w(b_1d_1)\right]+w(b_1)} q^{w(a_1d_1)(r-2)+w(a_1c_1)(c_1-2)+w(b_1d_1)(r-b_1-1)}$$

number of possibilities for the first row of burst error that falls into the generic class where $w(a) = 0$ if $a = 0$ and otherwise $w(a) = 1$. The same argument is still valid for the last row. Thus, for $j = 1$ and $j = p$, if

$$\begin{aligned}\gamma(X_j) = (q-1)^{2\left[w(a_1d_1)+w(a_1c_1)+w(b_1d_1)\right]+w(b_1)} \cdot \\ q^{w(a_1d_1)(r-2)+w(a_1c_1)(c_1-2)+w(b_1d_1)(r-b_1-1)} \cdot X_j^{\max\{rd_j, c_j, b_j, a_j\}}\end{aligned}$$

is substituted for the term $z_j^{a_j} x_j^{b_j} X_j^{c_j} y_j^{d_j}$ in the generic polynomial, then the new term expression gives all possible weight distributions for the jth row.

For $j = 2, \ldots p-1$, there are four cases to be considered. If the term $z_j y_j$ exists then, we substitute $(q-1)^2 q^{r-2} X_j^r$. If the term y_j exists only, then we substitute $(q-1)q^{r-2}X_j^r$. If the term z_j exists only, then we substitute $(q-1)X_j + \sum_{i=2}^{r-1} q^{i-2}(q-1)^2 X_j^i$. Finally, if none of z_j or y_j exist, then we substitute the term $1 + \sum_{i=2}^{r-1} q^{i-1}(q-1)X_j^i$. Hence, we obtain the multivariable polynomial $H_{p\times r}$ which is the spectra weight enumerator of burst error arrays of order $p \times r$ in $M_{p\times r}(R)$.

Thus we obtain the following theorem:

Theorem 5. *Let $G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y})$ be the generic polynomial. Then, for $j = 1$ and $j = p$, by substituting $\gamma(X_j)$ for $z_j^{a_j} x_j^{b_j} X_j^{c_j} y_j^{d_j}$ in G, and further substituting $(q-1)^2 q^{r-2} X_j^r$, $(q-1)q^{r-2} X_j^r$, and $(q-1)X_j + \sum_{i=2}^{r-1} q^{i-2}(q-1)^2 X_j^i$ for $z_j y_j$, y_j, and z_j and multiplying the terms that do not include the factors z_j and y_j by $1+\sum_{i=2}^{r-1} q^{i-1}(q-1)X_j^i$ for $j \neq 1$ and $j \neq p$, we obtain the spectra weight enumerator polynomial $H_{p\times r}(\tilde{X})$ in the space of burst errors of order $p \times r$.*

Example 5. *Let G be given as in Example 3. For instance, for $z_1 y_1 X_2^2$ and $z_2 y_2 X_1^2$ we substitute $2X_2^3 X_1^2$ respectively. Hence, by applying the necessary substitutions, we have*

$$H(X_1,X_2) = 4X_1^3 X_2 + 6X_1^3 X_2^2 + 12X_2^3 X_1^3 + 6X_2^3 X_1^2 + 4X_2^3 X_1.$$

Now, we can compare the terms of $H(X_1,X_2)$ with the list in the Example 3. For instance, the term $12X_2^3 X_1^3$ indicates that there are 12 burst errors of weight distribution $(3,3)$.

Example 6. *Let G be given as in Example 4. By applying Theorem 5, we get*

$$\begin{aligned} H(X_1,X_2,X_3) = & 28X_1^2 X_2^3 X_3^3 + 6X_1^2 X_3^3 + 12X_1^3 X_3^3 + 14X_1^2 X_2^2 X_3^3 \\ & + 28X_1^3 X_2^2 X_3^3 + 8X_1^2 X_2 X_3^3 + 4X_1 X_2 X_3^3 + 8X_1 X_2^2 X_3^3 \\ & + 16X_1 X_2^3 X_3^3 + 4X_1 X_3^3 + 56X_1^3 X_2^3 X_3^3 + 16X_1^3 X_2 X_3^3 \\ & + 14X_1^2 X_3^2 X_2^3 + 28X_1^3 X_3^2 X_2^3 + 8X_1^3 X_3^2 X_2 + 14X_1^3 X_3^2 X_2^2 \\ & + 8X_1 X_3^2 X_2^3 + 6X_1^3 X_3^2 + 4X_1 X_2^3 X_3 + 4X_1^3 X_2 X_3 + 4X_1^3 X_3 \\ & + 8X_1^2 X_2^3 X_3 + 8X_1^3 X_2^2 X_3 + 16X_1^3 X_2^3 X_3. \end{aligned}$$

Definition 10. *The polynomial, $B^{p\times r}(t) = \sum_{A\in\mathbb{B}} t^{w_N(A)} = \sum_{i=1}^{p\cdot r} b_i t^i$ is said to be the weight enumerator of bursts of order $p \times r$ in the space $M_{p\times r}(R)$.*

Corollary 2. *By substituting $X_i = t$ in $H(\tilde{X})$, we obtain $B_{p\times r}(t)$ which is the weight enumerator of bursts of order $p \times r$. Further, the $B_{p\times r}(R,w)$ numbers formulated in Theorem 2 by [5] are reobtained as follows:*

$$B_{p\times r}(R,w) = \sum_{i=1}^{w} b_i$$

where b_i's are coefficients of $B_{p\times r}(t)$.

We would like to emphasize that by applying Corollary 2, we are able to compute $B_{p\times r}(R,w)$ quite easily. In [5], for each weight w where $1 \leq w \leq pr$, the computation of $B_{p\times r}(R,w)$ has to be carried out separately by solving the necessary Diophantine inequalities (see Eq.(2)) and then applying the formula (see Eq. (1)).

Example 7. *The following generic polynomial G gives the term representation of $A \in M_{4\times 4}(F_2)$ generic burst errors of order 4×4 :*

$$G(\tilde{z}, \tilde{x}, \tilde{X}, \tilde{y}) = \sum_{(i,j,k,l)\in\{0,1\}^4} z_1^i y_1^j z_4^k y_4^l \overline{Z}_{ik}(\alpha) \overline{Y}_{jl}(\alpha) \gamma_{ij}(X_1) \gamma_{kl}(X_4).$$

Applying necessary substitutions given in Corollary 2, we obtain the weight enumerator of burst errors:

$$B_{4\times4}(t) = 3840t^{16} + 7680t^{15} + 9600t^{14} + 9728t^{13} + 8288t^{12} + 5856t^{11} + 3504t^{10} + 1824t^{9} + 792t^{8} + 272t^{7} + 72t^{6} + 16t^{5}.$$

Letting $t = 1$, in $B_{4\times4}(t)$ we have $B_{4\times4}(1) = 51472$. Further, $B_{4\times4}(F_2, 7) = 16+72+272 = 360$ which is the number of burst errors of type 4×4 of weight 7 or less.

3. Enumerating Bursts in Larger Space

In this section we consider burst of errors of order $p \times r$ in the space $M_{m\times s}(R)$ where $1 \le p < m$ and $1 \le r < s$.

Example 8. *We list two burst errors of order 3×3 in the space $M_{4\times4}(F_2)$.*

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lemma 2. *Let A be a burst of order $p \times r$ in the space $M_{m\times s}(R)$ where $1 \le p < m$ and $1 \le r < s$. If $T^{p\times r}_{p\times r}(R)$ is the number of burst errors of order $p \times r$ in the space $M_{p\times r}(R)$, then $T^{p\times r}_{m\times s}(R) = (s - r + 1)(m - p + 1)T^{p\times r}_{p\times r}(R)$ gives the number of burst of order $p \times r$ in the space $M_{m\times s}(R)$.*

Proof. By definition, a burst of order $p \times r$ in the space $M_{m\times s}(R)$ is a matrix of order $m \times s$ that includes a submatrix A which is a burst error of order $p \times r$ itself in the space $M_{p\times r}(R)$ and the entries that fall out of this submatrix are all zeroes. Hence, placing A as a submatrix of a matrix of size $m \times s$ is possible in $s - r + 1$ ways moving from the left to the right starting from the position $(1, 1)$ for both the matrix and the submatrix. Also, for a given possible position obtained above, there exist also $m - p + 1$ movements downwards for obtaining submatrices. Thus, there exist $(s - r + 1)(m - p + 1)$ ways that give raise to new submatrices for each burst error of order $p \times r$ in the space $M_{m\times s}(R)$. Therefore, the number of burst of order $p \times r$ in the space $M_{m\times s}(R)$ is $(s - r + 1)(m - p + 1)T^{p\times r}_{p\times r}(R)$.

Hence, it is possible to construct all burst errors of size $p \times r$ in $M_{m\times s}(R)$ knowing all burst errors of size $p \times r$ in $M_{p\times r}(R)$. This can be done easily by positioning the matrices of size $p \times r$ as sub matrices of matrices of size $m \times s$. Since after repositioning, the RT weights will change, the main problem is to control the weights of burst errors when repositioning is done. If we know the weight spectra polynomial $H(X)$ of burst errors of size $p \times r$ in $M_{p\times r}(R)$, then by multiplying $H(X)$ with suitable variables or in other words applying a translation map, we can obtain the weight spectra of burst errors of size $p \times r$ in $M_{p\times r}(R)$ as explained below.

Definition 11. *Let*

$$H(X_1, X_2, \dots, X_p) = \sum_{(i_1, i_2, \dots, i_p) \in \mathbb{N}^p} h(i_1, i_2, \dots, i_p) X_1^{i_1} X_2^{i_2} \cdots X_p^{i_p}$$

be the weight spectra polynomial where $h(i_1, i_2, \dots, i_p) \in \mathbb{N}$ *and* $\mathbb{N}$ *is the set of natural numbers. Then, we define a translation map*

$$T(H) = \sum_{(i_1, i_2, \dots, i_p) \in \mathbb{N}^p} h(i_1, i_2, \dots, i_p) X_1^{w(i_1)(i_1+1)} X_2^{w(i_2)(i_2+1)} \cdots X_p^{w(i_p)(i_p+1)}.$$

Lemma 3. *Let* $H(X)$ *be the weight spectra polynomial of burst of errors of order* $p \times r$ *in the space* $M_{p\times r}(R)$. *The weight spectra polynomial of burst of errors of order* $p \times r$ *in the space* $M_{m\times s}(R)$ *is given by*

$$W(\tilde{X}) = (m - p + 1) \sum_{i=0}^{s-r+1} T^i(H).$$

Proof. Polynomial H gives the terms that correspond to burst errors of order $p \times r$ whose $(1,1)$ entry position is located at $(1,1)$. The polynomial $T(H)$ gives the terms that correspond to burst errors whose $(1,1)$ entry position is located at $(1,2)$. There are $s - r + 1$ translations possible for a submatrix of order $p \times r$ in the space of matrices of order $m \times s$. The sum of all these give all possible submatrices as burst errors of order $p \times r$ whose first rows are situated as first rows of burst error of order $m \times s$. For each submatrix obtained above, we can move it downwards to obtain all possible burst errors. This is possible in $m - p + 1$ ways for each case.

Definition 12.

$$W_{m\times s}^{p\times r}(t) = \sum_{i=0}^{ms} w_i t^i$$

is called the burst weight enumerator of burst errors of order $p \times r$ *in the space* $M_{m\times s}(R)$.

Now, it is clear that by setting $X_1 = X_2 = \cdots = X_p = t$ in $W(\tilde{X})$ we obtain the burst weight enumerator $W_{m\times s}^{p\times r}(t) = \sum_{i=0}^{ms} w_i t^i$.

Example 9. *The weight spectra polynomial of burst errors of order* 2×3 *in the space of matrices* $M_{3\times 5}(F_2)$ *is computed in the following way. First, the weight spectra polynomial of burst of errors of order* 2×3 *in the space of matrices* $M_{2\times 3}(F_2)$ *is already computed in Example 3. Next, by Lemma 3 by applying* $T(H)$ *and* $T^2(H)$, *we get all possible weight distributions of the terms that correspond to burst errors.*

$$\begin{aligned} H + T(H) + T^2(H) = {} & 3X_1^3X_2^5 + 4X_1^5X_2^4 + 12X_1^5X_2^5 + 5X_1^4X_2^5 + 6X_1^5X_2^3 \\ & + X_1^3X_2^7 + 2X_1^7X_2^5 + X_1^4X_2^7 + 3X_1^2X_2^4 + 4X_1^4X_2^3 \\ & + 12X_1^4X_2^4 + 5X_1^3X_2^4 + 4X_1^4X_2^2 + X_1^2X_2^6 + 2X_1^6X_2^4 \end{aligned}$$

$$+X_1^3X_2^6+3X_1X_2^3+4X_2^2X_1^3+12X_1^3X_2^3+5X_2^3X_1^2$$
$$+4X_2X_1^3+X_1X_2^5+X_2^5X_1^2.$$

Setting $X = Y = t$ and multiplying by $p-r+1$ in $W(\tilde{X}) = 2(H + T(H) + T^2(H))$, we obtain

$$W_{3\times 5}^{2\times 3}(t) = 4t^{12} + 2t^{11} + 30t^{10} + 20t^9 + 44t^8 + 20t^7 + 40t^6 + 18t^5 + 14t^4.$$

Hence, the number of burst errors of ρ-weight 3 or less is equal to $T_{3\times 3}^{2\times 2}(F_2, 3) = 10 + 6 + 2$.

Corollary 3. *By substituting $X_i = t$ in $W(\tilde{X})$, we obtain $W_{m\times s}^{p\times r}(t)$ which is the weight enumerator of bursts of order $p \times r$ in $M_{m\times s}(R)$. Further,*

$$B_{m\times s}^{p\times r}(R, w) = \sum_{i=1}^{w} w_i$$

where w_i's are the coefficients of $W_{m\times s}^{p\times r}(t)$.

4. Some Applications

From previous sections we have presented a constructive method for computing $B_{m\times s}^{p\times r}(R)$ and $B_{m\times s}^{p\times r}(R, w)$. By making use of these results we have the following theorems:

Theorem 6. *Let C be an array code over R. If C corrects all burst errors of type $p \times r$, then*

$$\frac{|R|^{ms}}{|C|} \geq 1 + B_{m\times s}^{p\times r}(R) \tag{3}$$

where "$|\cdot|$" stands for the cardinality of the set.

Proof. C is an abelian additive group. If C corrects all burst errors of type $p \times r$, then all these bursts must fall into different cosets. So the number of cosets including C itself should be larger or equal to the number of burst errors plus one which stands for zero codeword.

A similar argument that depends on syndrome decoding leads to the following theorem:

Theorem 7. *Let C be an array code over R. If C corrects all burst errors of type $p \times r$ and RT-weight equal to w or less, then*

$$\frac{|R|^{ms}}{|C|} \geq 1 + \sum_{i=1}^{p}\sum_{j=1}^{r} B_{m\times s}^{i\times j}(R, w). \tag{4}$$

Burst error correction is based on assumption of errors occurring nearby and sometimes in particular predesigned places. Reiger [6] proved an inequality by assuming predesigned burst errors (burst errors occurring in the last consecutive digits) for block codes over fields. Later, Jain in [5] extended these results for array codes over finite fields. Here, we extend them further for array codes over finite commutative rings.

Theorem 8 ((A Reiger's Type Bound)). *Let C be an array code over R with no burst errors of type $p \times r$ or smaller in a predesigned place of size $p \times r$. Then,*

$$\frac{|R|^{ms}}{|C|} \geq q^{pr}. \tag{5}$$

Proof. Without loss of generality, we may assume that the predesigned place is the first p rows and the first r columns of size $p \times r$. In other words we assume that the burst errors of size may appear only in this place. Two different bursts of this type does not fall into the same coset. Otherwise, If two different burst errors of this type fall into the same coset, then their the difference will be a codeword in the code. This will lead to a contradiction since the array code does not contain such burst errors. Hence, the number of cosets must be larger or equal to the number of all possible burst errors of type $p \times r$ which equals to q^{pr}.

5. Conclusion

A new approach that avoids solving integer inequalities on enumerating burst errors of arrays is presented. This method is applied to array codes over finite rings which is a generalization of previous results. Finally, some applications of the results are presented .

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