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Blow-up for nonlinear heat equations with absorptions

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Abstract. This paper deals with the blow-up of positive solution of the nonlinear heat equation $u_t = \nabla(a(u)\nabla u) - f(u)$ subject to nonlinear boundary condition $\frac{\partial u}{\partial n} = b(u)$. Under suitable assumptions on nonlinear functions a, f, b and initial data $u_0(x)$, we obtain the blow-up rate and the blow-up set of the solutions of the problem by the Nirenberg maximum principle.

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Key words: Nonlinear heat equation, Nonlinear boundary condition, Blow-up, Absorption, Maximum principle.

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1. Introduction

In the last few decades, blow-up phenomena for the nonlinear parabolic equations with heat sources have been studied by many authors, and the reader is referred to [3-5, 9, 11, 14] and the references therein. For the parabolic equations with no sources, some necessary conditions for the global existence and blow-up of the solutions are given in [6, 8, 12-13]. Here we are interested in the blow-up phenomena of the solutions of the parabolic problems with absorptions at interior of the domains.

In this paper, we investigate the following initial-boundary value problem:

$$u_t = \mathrm{div}(a(u)\nabla u) - f(u) \quad \text{in } \Omega \times (0, T) \tag{1}$$

$$\frac{\partial u}{\partial n} = b(u) \ \text{ on } \ \partial\Omega \times (0, T) \tag{2}$$

$$u(x, 0) = u_0(x) > 0 \quad \text{in } \Omega \tag{3}$$

where Ω is a bounded domain in R^N with C^2 boundary, $\frac{\partial u}{\partial n}$ denotes the outward normal derivative; $u_0(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$ is a positive function, satisfying compatibility condition; $a(\cdot), f(\cdot), b(\cdot)$ are smooth positive functions.

Problem (1)-(3) has been formulated from physical models arising in various fields of applied sciences. For example it can be interpreted as a heat conduction problem with nonlinear

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diffusivity, absorption at interior of the domain and a nonlinear radiation law on the boundary of the material body.

The local existence and uniqueness of positive classical solution of the problem (1)-(3) were established by Amann [1], finite extinction time is studied by Leung and Zhang [7]. Here we are interested in the blow-up phenomenon of the solution of the problem (1)-(3).We say that the solution u blows up if there exists a $0 < T < +\infty$ such that $\lim_{t\to T} \| u \|_{L^\infty(\Omega)} = +\infty$.

In this paper, we study the blow-up rate and blow-up set by the maximum principles. Our results generalize and deepen ones from corresponding work in [2-6, 8-12].

2. Blow-up rate and blow-up set

Theorem 1. *Let $u(x,t)$ be a solution of the problem (1)-(3). Assume that:*
(1) $\int_0^{+\infty} \frac{a(s)}{b(s)} ds < +\infty$ *and* $a'(s), (\frac{b'(s)}{a(s)})', (\frac{b(s)}{f(s)})' \geq 0$ *for* $s > 0$,
(2) $\operatorname{div}(a(u_0)\nabla u_0) \geq f(u_0)$.
Then $u(x,t)$ must blow up in finite time T and there exists a constant $\delta > 0$ such that

$$\sup_{x\in\overline{\Omega}} u(x,t) \leq H(\delta(T-t)) \quad \text{for } 0 < t < T,$$

where $H = G^{-1}$ *and* $G(s) = \int_s^{+\infty} \frac{a(\mu)}{b(\mu)} d\mu$.

Proof. We will use the ideas of [4] or [14].

Step 1: Growth estimate.
Introduce an auxiliary function

$$J(x,t) = a(u)u_t - \delta b(u) \qquad (\delta > 0) \tag{4}$$

then we have

$$\nabla J = a' u_t \nabla u + a \nabla u_t - \delta b' \nabla u$$

$$\triangle J = a\triangle u_t + 2a' \nabla u \nabla u_t + a'' u_t \mid \nabla u \mid^2 + a' u_t \triangle u - \delta b'' \mid \nabla u \mid^2 - \delta b' \triangle u$$

and

$$J_t = a' u_t^2 + a^2 \triangle u_t + 2aa' \nabla u \nabla u_t + aa'' u_t \mid \nabla u \mid^2 + aa' u_t \triangle u - (af' + \delta b')u_t.$$

Hence,

$$J_t - a\triangle J + f' J = a' u_t^2 + \delta(ab'' - a' b') \mid \nabla u \mid^2 + \delta(b'f - bf'). \tag{5}$$

Under the assumptions of theorem 1, we obtain

$$J_t - a\triangle J + f' J \geq 0. \tag{6}$$

Since $u_0(x) > 0$ and $\operatorname{div}(a(u_0)\nabla u_0) \geq f(u_0)$, by the Nirenberg maximum principle, $u \geq 0$ and $u_t \geq c$ in $\overline{\Omega} \times (\varepsilon_0, T)$, where c, ε_0 are some positive constants. It following that

$$J(x,\varepsilon_0) = a(u(x,\varepsilon_0))u_t(x,\varepsilon_0) - \delta b(u(x,\varepsilon_0)) \geq 0 \tag{7}$$

provided δ is small enough. Note that

$$\frac{\partial u}{\partial n} = b(u) \text{ on } \partial\Omega \times (0, T),$$

we have

$$\frac{\partial u_t}{\partial n} = b'(u)u_t \text{ on } \partial\Omega \times (0, T).$$

Thus

$$\frac{\partial J}{\partial n} = a\frac{\partial u_t}{\partial n} + (a'u_t - \delta b')\frac{\partial u}{\partial n} = (ab)'u_t - \delta bb'. \tag{8}$$

Using (4) and (8) we have

$$\frac{\partial J}{\partial n} - \frac{(ab)'}{a}J = \frac{\delta a'b^2}{a} \geq 0 \text{ on } \partial\Omega \times (\varepsilon_0, T). \tag{9}$$

Thus, by the maximum principle for parabolic problems, $J(x, t) \geq J(x, \varepsilon_0) \geq 0$ in $\Omega \times (\varepsilon_0, T)$, i.e.,

$$u_t \geq \delta\frac{b(u)}{a(u)} \quad \text{ in } \Omega \times (\varepsilon_0, T). \tag{10}$$

Step 2: Blow-up rate.

Set

$$G(s) = \int_s^\infty \frac{a(s)}{b(s)}ds$$

then

$$-(G(u))_t = \frac{a(u)}{b(u)}u_t.$$

By the growth estimate (10) we have

$$-(G(u))_t \geq \delta \quad \text{ in } \Omega \times (\varepsilon_0, T).$$

By integration from t to T

$$G(u(x, t)) - G(u(x, T)) \geq \delta(T - t) \quad \varepsilon_0 < t < T$$

therefore also

$$G(u(x, t)) \geq \delta(T - t). \tag{11}$$

Since $\int_0^{+\infty} \frac{a(s)}{b(s)}ds < +\infty$, by (11), $u(x, t)$ must blow up in finite time T and

$$\sup_{x\in\overline{\Omega}} u(x, t) \leq G^{-1}(\delta(T - t)) \quad \text{for } 0 < t < T.$$

The proof of Theorem 1 is completed.

With analogy to Theorem 1, one can also obtain bounds for the global solutions.

Corollary 1. *Let $u(x,t)$ be a solution of the problem (1)-(3). If $\int_0^{+\infty} \frac{a(s)}{b(s)} ds = +\infty$, $a'(s) \le 0$, and $(\frac{b'(s)}{a(s)})' \le 0$ for $s > 0$, then $u(x,t)$ exists globally and*

$$\sup_{x \in \overline{\Omega}} u(x,t) \le G^{-1}((t + G(M))) \quad \text{for} \quad t > 0,$$

where $M = \max_{\overline{\Omega}} u_0$.

Proof. Let $w(x,t)$ be a smooth positive solution of the following problem:

$$w_t = \operatorname{div}(a(w)\nabla w) \quad \text{in } \Omega \times (0,T), \tag{12}$$

$$\frac{\partial w}{\partial n} = b(w) \ \text{ on } \ \partial\Omega \times (0,T), \tag{13}$$

$$u(x,0) = M = \max_{\overline{\Omega}} u_0 \quad \text{in } \Omega. \tag{14}$$

With analogy to the proof of Theorem 1, by the maximum principle we have $L = a(w)w_t - b(w) \le 0$, i.e.,

$$w_t \le \frac{b(w)}{a(w)} \quad \text{in } \Omega \times (0,T). \tag{15}$$

For each fixed $x \in \overline{\Omega}$, we get by integration (15)

$$\int_M^{w(x,t)} \frac{a(s)}{b(s)} ds \le t. \tag{16}$$

It follows from assumptions that $w(x,t)$ must be a global solution. With inequality (16), one gets

$$G(w(x,t)) - G(M) = \int_C^{w(x,t)} \frac{d\beta(s)}{f(s)} \le t$$

and

$$w(x,t) \le G^{-1}(t + G(M)).$$

By the comparison principle we know that $w(x,t)$ is an upper solution of (1)-(3). Thus

$$u(x,t) \le w(x,t) \quad \text{in } \Omega \times (0,T).$$

The proof of Corollary 1 is complete.

We shall prove in the following theorem that the blowup will occur only at the boundary of the domain.

Theorem 2. *Suppose that the assumptions of Theorem 1 hold, and there exists a positive constant C_0 such that $s(\frac{b}{a})'(H(s)) \le C_0$ for $s > 0$. Then for any $\Omega' \subset\subset \Omega$ and $\varepsilon_0 > 0$,*

$$\sup_{x \in \Omega',\ t \in [\varepsilon_0, T)} u(x,t) < +\infty.$$

Proof. We will use the ideas of [6]. Let $d(x) = \text{dist}(x, \partial\Omega)$ and

$$v(x) = d^2(x) \ \text{ for } x \in N_\varepsilon(\partial\Omega)$$

where $N_\varepsilon(\partial\Omega) = \{x \in \Omega : d(x) < \varepsilon\}$. Since $\partial\Omega$ is C^2, the function $v(x)$ is in $C^2(\overline{N_\varepsilon(\partial\Omega)})$ if ε is small enough. Therefore, there exists a constant $C > 0$ such that

$$\text{div}(a(v)\nabla v) - \frac{C_0}{v}|\nabla v|^2 \geq -C \quad \text{in } \overline{N_{\varepsilon_0}(\partial\Omega)}$$

if ε_0 is small enough. We next extend $v(x)$ to a function on $\overline{\Omega}$ such that $v \in C^2(\overline{\Omega})$ and $v \geq c_0 > 0$ on $\overline{\Omega/N_{\varepsilon_0}(\partial\Omega)}$. Then

$$\text{div}(a(v)\nabla v) - \frac{C_0}{v}|\nabla v|^2 \geq -C^* \quad \text{on } \overline{\Omega}$$

for some $1 \geq C^* > 0$.

Set $w(x,t) = C_1 H(\tau)$, where $\tau = \delta(v(x) + C^*(T-t))$ and $C_1 > 0$. Then

$$w_t - \text{div}(a(w)\nabla w) + f(w) \geq 0 \quad \text{in } \Omega \times (\varepsilon_0, T).$$

By Theorem 1 we have

$$\sup_{x \in \overline{\Omega}} u(x,t) \leq H(\delta(T-t)) \quad \text{on } \partial\Omega \times (\varepsilon_0, T).$$

Thus

$$w(x,t) = C_1 H(\delta C^*(T-t))) > H(\delta(T-t)) \geq u(x,t) \quad \text{on } \partial\Omega \times (\varepsilon_0, T)$$

if $C_1 > 1$. Take C_1 to be large enough so that $w(x,\varepsilon_0) \geq u(x,\varepsilon_0)$. Then the maximum principle implies that $w(x,t) \geq u(x,t)$ in $\Omega \times (\varepsilon_0, T)$. Therefore for $\Omega' \subset\subset \Omega$

$$u(x,t) \leq C_1 H(\delta(v(x) + C^*(T-t))) \leq C_1 H(\delta v(x))$$

i.e.,

$$\sup_{x \in \Omega',\ t \in [\varepsilon_0, T)} u(x,t) < +\infty.$$

The theorem 2 is proved.

In our theorems, if $f(u) \equiv 0$, $a(u) \equiv 1$ and $b(u) = u^p$ $(p > 1)$, then the following conclusion holds:

Corollary 2. *Let $u(x,t)$ be a smooth solution of the following problem:*

$$\begin{cases} u_t = \triangle u & \text{in } \Omega \times (0,T) \\ \frac{\partial u}{\partial n} = u^p & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) > 0 & \text{in } \Omega. \end{cases}$$

If $\triangle u_0 \geq 0$, then $u(x,t)$ blows up in finite time and blowup will occur only at the boundary of the domain.

This is the case of [6].

3. Concluding remarks and applications

Problem (1)-(3) arises in the nonlinear diffusion process, in which $\mathrm{div}(a(u)\nabla u)$ denotes the nonlinear diffusion effect, $f(u)$ denotes absorption in the interior of the domain, and $\frac{\partial u}{\partial n}$ denotes the boundary flux along the outward normal direction to the domain. With this model, all of the results obtained in the preceding sections are physically meaningful.

Our results show that the strength of the boundary flux plays a key role in the blowup properties of the problem (1)-(3). If the boundary flux is sufficiently strong, then it will bring about blowup in a finite time, and the blowup will occur only at the boundary of the domain. If the boundary flux is not sufficiently strong, it is probable that the solution may never blow up.

As the application of theorems, now we consider the following porous medium problem

$$u_t = \triangle u^p - u^q \qquad \text{in } \Omega \times (0,T),$$

$$\frac{\partial u}{\partial n} = u^r \quad \text{on } \partial\Omega \times (0,T),$$

$$u(x,0) = u_0(x) > 0 \quad \text{in } \Omega,$$

where $p, q, r > 0$.

If $r \le p \le 1$, then Corollary 1 hold, every positive solution $u(x,t)$ of the problem exists globally. If $r \ge \max\{1, p, q\}$ and $\triangle u_0^p \ge u_0^q$, by Theorems 1-2 we know that every positive solution $u(x,t)$ of the problem blows up in a finite time T and the blowup will occur only at the boundary of the domain. Moreover, there exits a constant $C > 0$ such that

$$\sup_{x\in\overline{\Omega}} u(x,t) \le \frac{C}{(T-t)^{\frac{1}{r-p}}} \qquad \varepsilon_0 < t < T,\ \varepsilon_0 > 0.$$

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