EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 3, 2012, 317-332
ISSN 1307-5543 – www.ejpam.com

Bivariate Generalization of The Inverted Hypergeometric Function Type I Distribution

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Abstract. The bivariate inverted hypergeometric function type I distribution is defined by the probability density function proportional to $x_1^{\nu_1-1}x_2^{\nu_2-1}\left(1+x_1+x_2\right)^{-(\nu_1+\nu_2+\gamma)}\,{}_2F_1(\alpha,\beta;\gamma;(1+x_1+x_2)^{-1})$, $x_1>0$, $x_2>0$, where ν_1, ν_2, α, β and γ are suitable constants. In this article, we study several properties of this distribution and derive density functions of X_1/X_2, $X_1/(X_1+X_2)$ and X_1+X_2. We also consider several products involving bivariate inverted hypergeometric function type I, beta type I, beta type II, beta type III, Kummer-beta and hypergeometric function type I variables.

2010 Mathematics Subject Classifications: 62E15, 60E05

Key Words and Phrases: Appell's first hypergeometric function, Beta distribution, Gauss hypergeometric function, Humbert's confluent hypergeometric function, product, transformation.

1. Introduction

The random variable X is said to have an inverted hypergeometric function type I distribution, denoted by $X\sim IH^I(\nu,\alpha,\beta,\gamma)$, if its probability density function (p.d.f.) is given by Nagar and Alvarez [8],

$$\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)}\frac{x^{\nu-1}}{(1+x)^{\nu+\gamma}}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x}\right),\quad x>0, \tag{1}$$

where $\nu>0$, $\gamma>0$, $\gamma+\nu>\alpha+\beta$, and ${}_2F_1$ is the Gauss hypergeometric function. For $\alpha=\gamma$, the density (1) reduces to a beta type II density given by

$$\frac{\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu-\beta)}\frac{x^{\nu-\beta-1}}{(1+x)^{\nu-\beta+\gamma}},\quad x>0,$$

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and for $\beta = \gamma$, the inverted hypergeometric function type I density slides to

$$\frac{\Gamma(\gamma+\nu-\alpha)}{\Gamma(\gamma)\Gamma(\nu-\alpha)}\frac{x^{\nu-\alpha-1}}{(1+x)^{\nu-\alpha+\gamma}}, \quad x>0.$$

Further, for $\alpha = 0$ or $\beta = 0$, the inverted hypergeometric function type I density simplifies to a beta type II density with parameters ν and γ.

Recently, Nagar and Alvarez [8] studied several properties and stochastic representations of the inverted hypergeometric function type I distribution. Zarrazola and Nagar [16] derived the density function of the product of two independent random variables having inverted hypergeometric function type I distribution. They also derive densities of several other products involving hypergeometric function type I, beta type I, beta type II, beta type III, Kummer-beta and hypergeometric function type I variables.

The bivariate generalization of the inverted hypergeometric function type I distribution, denoted by $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, is defined by the density [Nagar, Bran-Cardona and Gupta 9],

$$C(\nu_1,\nu_2;\alpha,\beta,\gamma)\frac{x_1^{\nu_1-1}x_2^{\nu_2-1}}{(1+x_1+x_2)^{\nu_1+\nu_2+\gamma}}\,{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right), \tag{2}$$

where $x_1 > 0$, $x_2 > 0$, and $C(\nu_1, \nu_2; \alpha, \beta, \gamma)$ is the normalizing constant given by

$$C(\nu_1,\nu_2;\alpha,\beta,\gamma) = \frac{\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)},$$

with $\nu_1 > 0, \nu_2 > 0$, $\gamma > 0$, and $\nu_1 + \nu_2 + \gamma > \alpha + \beta$.

For $\alpha = 0$ or $\beta = 0$, the density (2) slides to a Dirichlet type II density of order 3 with parameters ν_1, ν_2 and γ.

It can also be observed that bivariate generalization of the hypergeometric function type I distribution defined by the density (2) belongs to the Liouville family of distributions proposed by Marshall and Olkin [6] and Sivazlian [14].

In this article we study several properties of the bivariate generalization of the hypergeometric function type I distribution defined by the density (2).

In Section 2, we derive results such as the marginal and the conditional densities, moments and correlation and in Section 3 we show that if $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, then, $X_1 + X_2 \sim H^I(\nu_1 + \nu_2, \alpha, \beta, \gamma)$, which is independent of $X_1/(X_1 + X_2) \sim B^I(\nu_1, \nu_2)$ and $X_1/X_2 \sim B^{II}(\nu_1, \nu_2)$. In Section 4, we derive density functions of (X_1X_3, X_2X_3), where (X_1, X_2) and X_3 are independent, $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$ and

(i) $X_3 \sim IH^I(\kappa, \mu, \rho, \sigma)$,

(ii) $X_3 \sim B^{II}(\kappa, \sigma)$,

(iii) $X_3 \sim KB(\kappa, \mu, \lambda)$

(iv) $X_3 \sim B^I(\kappa, \mu)$,

(v) $X_3 \sim B^{III}(\kappa,\mu)$, and

(vi) $X_3 \sim H^{I}(\kappa,\mu,\rho,\sigma)$.

Finally, in appendix we give definitions and results on Gauss hypergeometric function, Appell's first hypergeometric function F_1, Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$ and statistical distributions.

2. Properties

In this section we study several properties of the bivariate distribution defined in Section 1. We first derive marginal and conditional distributions.

Theorem 1. *If $(X_1,X_2) \sim IH^{I}(\nu_1,\nu_2;\alpha,\beta,\gamma)$, then the p.d.f. of X_1 is given by*

$$\frac{\Gamma(\nu_1+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\gamma)\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)} \times\frac{x_1^{\nu_1-1}}{(1+x_1)^{\nu_1+\gamma}}{}_3F_2\left(\alpha,\beta,\nu_1+\gamma;\gamma,\nu_1+\nu_2+\gamma;\frac{1}{1+x_1}\right),\quad x_1>0. \tag{3}$$

Proof. By integrating x_2 in (2), we get the marginal p.d.f. of X_1 as

$$C(\nu_1,\nu_2;\alpha,\beta,\gamma)\frac{x_1^{\nu_1-1}}{(1+x_1)^{\nu_1+\gamma}}\int_0^1 z^{\nu_1+\gamma-1}(1-z)^{\nu_2-1}{}_2F_1\left(\alpha,\beta;\gamma;\frac{z}{1+x_1}\right)\mathrm{d}z,$$

where we have used the substitution $z=(1+x_1)/(1+x_1+x_2)$. Now, the desired result is obtained by using (A.2).

For $\alpha=\gamma$, the density (2) reduces to

$$\frac{\Gamma(\nu_1+\nu_2)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu_1+\nu_2-\beta)}\frac{x_1^{\nu_1-1}x_2^{\nu_2-1}}{(x_1+x_2)^{\beta}\,(1+x_1+x_2)^{\nu_1+\nu_2+\gamma-\beta}}, \tag{4}$$

where $x_1>0$ and $x_2>0$. The marginal density of X_1 in this case is given by

$$\frac{\Gamma(\nu_1+\gamma)\Gamma(\nu_1+\nu_2)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\gamma)\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2-\beta)} \times\frac{x_1^{\nu_1-1}}{(1+x_1)^{\nu_1+\gamma}}{}_2F_1\left(\beta,\nu_1+\gamma;\nu_1+\nu_2+\gamma;\frac{1}{1+x_1}\right),\quad x_1>0. \tag{5}$$

Using the above theorem, the conditional density function of X_1 given $X_2=x_2>0$ is obtained as

$$\frac{\Gamma(\gamma+\nu_1+\nu_2)}{\Gamma(\nu_1)\Gamma(\gamma+\nu_2)}\frac{x_1^{\nu_1-1}(1+x_2)^{\gamma+\nu_2}}{(1+x_1+x_2)^{\gamma+\nu_1+\nu_2}}\frac{{}_2F_1(\alpha,\beta;\gamma;(1+x_1+x_2)^{-1})}{{}_3F_2(\alpha,\beta,\gamma+\nu_2;\gamma,\gamma+\nu_1+\nu_2;(1+x_2)^{-1})},$$

where $x_1 > 0$ and $x_2 > 0$. Further, using (2), the joint (r,s)-th moment is obtained as

$$E(X_1^r X_2^s) = C(\nu_1,\nu_2;\alpha,\beta,\gamma)\int_0^\infty\int_0^\infty \frac{x_1^{\nu_1+r-1}x_2^{\nu_2+s-1}}{(1+x_1+x_2)^{\nu_1+\nu_2+\gamma}}\times {}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right)\mathrm{d}x_2\,\mathrm{d}x_1.$$

Now, substituting $u = x_1/(x_1+x_2)$, $v = x_1+x_2$ and $z = 1/(1+v)$ with the Jacobian $J(x_1,x_2 \to u,z) = J(x_1,x_2 \to u,v)J(v \to z) = (1-z)/z^3$ in the above integral, one obtains

$$E(X_1^r X_2^s) = C(\nu_1,\nu_2;\alpha,\beta,\gamma)B(\nu_1+r,\nu_2+s)\int_0^1 z^{\gamma-r-s-1}(1-z)^{\nu_1+\nu_2+r+s-1}{}_2F_1(\alpha,\beta;\gamma;z)\,\mathrm{d}z.$$

Finally, evaluating the above integral using (A.2) and simplifying the resulting expression, we get

$$\begin{aligned} E(X_1^r X_2^s) &= \frac{\Gamma(\nu_1+r)\Gamma(\nu_2+s)\Gamma(\gamma-r-s)\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)} \\ &\quad \times {}_3F_2(\alpha,\beta,\gamma-r-s;\gamma,\nu_1+\nu_2+\gamma;1), \end{aligned}$$

where $\nu_1 + r > 0$, $\nu_2 + s > 0$ and $\gamma > r + s$. Now, substituting appropriately, we obtain

$$\begin{aligned} E(X_i) &= \frac{\nu_i}{\gamma-1}\frac{\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)} \\ &\quad \times {}_3F_2(\alpha,\beta,\gamma-1;\gamma,\nu_1+\nu_2+\gamma;1), \end{aligned}$$

$$\begin{aligned} E(X_i^2) &= \frac{\nu_i(\nu_i+1)}{(\gamma-1)(\gamma-2)}\frac{\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)} \\ &\quad \times {}_3F_2(\alpha,\beta,\gamma-2;\gamma,\nu_1+\nu_2+\gamma;1), \end{aligned}$$

and

$$\begin{aligned} E(X_1X_2) &= \frac{\nu_1\nu_2}{(\gamma-1)(\gamma-2)}\frac{\Gamma(\nu_1+\nu_2+\gamma-\alpha)\Gamma(\nu_1+\nu_2+\gamma-\beta)}{\Gamma(\nu_1+\nu_2+\gamma)\Gamma(\nu_1+\nu_2+\gamma-\alpha-\beta)} \\ &\quad \times {}_3F_2(\alpha,\beta,\gamma-2;\gamma,\nu_1+\nu_2+\gamma;1). \end{aligned}$$

Using $E(X_i)$, $E(X_i^2)$ and $E(X_1X_2)$, the expressions for $\mathrm{Var}(X_i)$, $\mathrm{Cov}(X_1,X_2)$ and $\mathrm{Corr}(X_1,X_2)$ can easily be calculated.

The stress-strength model describes the life of a component which has a random strength X_2 and is subjected to a random stress X_1. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $X_2 > X_1$. Thus, $R = Pr(X_1 < X_2)$ is a measure of the component reliability. In a recent paper, Nadrajah [7] has give an extensive survey on applications and computation of R when X_1 and X_2 follows bivariate distribution with dependence between them. If (X_1,X_2) has a bivariate inverted hypergeometric function type I distribution, then

$$R = C(\nu_1,\nu_2;\alpha,\beta,\gamma)\int_0^\infty x_1^{\nu_1-1}\int_{x_1}^\infty \frac{x_2^{\nu_2-1}}{(1+x_1+x_2)^{\nu_1+\nu_2+\gamma}}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right)\mathrm{d}x_2\,\mathrm{d}x_1.$$

Replacing ${}_2F_1\left(\alpha,\beta;\gamma;(1+x_1+x_2)^{-1}\right)$ by its series representation, we get

$$R = C(\nu_1,\nu_2;\alpha,\beta,\gamma)\sum_{i=0}^{\infty}\frac{(\alpha)_i(\beta)_i}{(\gamma)_i\, i!}\int_0^{\infty} x_1^{\nu_1-1}\int_{x_1}^{\infty}\frac{x_2^{\nu_2-1}}{(1+x_1+x_2)^{\nu_1+\nu_2+\gamma+i}}\,\mathrm{d}x_2\,\mathrm{d}x_1.$$

Now, using (A.8), we have

$$\begin{aligned} R &= C(\nu_1,\nu_2;\alpha,\beta,\gamma)\sum_{i=0}^{\infty}\frac{(\alpha)_i(\beta)_i}{(\nu_1+\gamma+i)\,(\gamma)_i\, i!} \\ &\quad\times\int_0^{\infty} x_1^{-(\gamma+i+1)}{}_2F_1\left(\nu_1+\nu_2+\gamma+i,\nu_1+\gamma+i;\nu_1+\gamma+i+1;-\frac{1+x_1}{x_1}\right)\mathrm{d}x_1. \end{aligned}$$

Finally, using (A.6), expanding ${}_2F_1$ in series form and using (A.7), we obtain

$$\begin{aligned} R &= C(\nu_1,\nu_2;\alpha,\beta,\gamma)\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_i(\beta)_i\,(\nu_1+\nu_2+\gamma+i)_k}{(\nu_1+\gamma+i)(\gamma)_i(\nu_1+\gamma+i+1)_k\, i!} \\ &\quad\times\frac{\Gamma(\nu_1+\nu_2)\Gamma(\gamma+i)}{\Gamma(\nu_1+\nu_2+\gamma+i)}{}_2F_1\left(\nu_1+\nu_2+\gamma+i+k,\nu_1+\nu_2;\nu_1+\nu_2+\gamma+i;-1\right). \end{aligned}$$

3. Distributions of Sum and Quotients

It is well known that if $(X_1,X_2)\sim D^{II}(\nu_1,\nu_2;\nu_3)$, then X_1/X_2 and $X_1/(X_1+X_2)$ are independent of X_1+X_2. Further, $X_1/X_2\sim B^{II}(\nu_1,\nu_2)$, $X_1/(X_1+X_2)\sim B^{I}(\nu_1,\nu_2)$, and $X_1+X_2\sim B^{II}(\nu_1+\nu_2,\nu_3)$. In this section we derive similar results when X_1 and X_2 have a bivariate inverted hypergeometric function type I distribution.

Theorem 2. *Let $(X_1,X_2)\sim IH^{I}(\nu_1,\nu_2;\alpha,\beta,\gamma)$. Then, $Z=X_1/(X_1+X_2)$ and $S=X_1+X_2$ are independent, $Z\sim B^{I}(\nu_1,\nu_2)$ and $S\sim IH^{I}(\nu_1+\nu_2,\alpha,\beta,\gamma)$.*

Proof. Transforming $Z=X_1/(X_1+X_2)$ and $S=X_1+X_2$ with the Jacobian $J(x_1,x_2\to z,s)=s$ in (2), we obtain the joint p.d.f. of Z and S as

$$C(\nu_1,\nu_2;\alpha,\beta,\gamma)z^{\nu_1-1}(1-z)^{\nu_2-1}\frac{s^{\nu_1+\nu_2-1}}{(1+s)^{\nu_1+\nu_2+\gamma}}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+s}\right),$$

where $0<z<1$ and $s>0$. Now, from the above factorization it is clear that Z and S are independent, $Z\sim B^{I}(\nu_1,\nu_2)$ and $S\sim IH^{I}(\nu_1+\nu_2,\alpha,\beta,\gamma)$.

Corollary 1. *Let $(X_1,X_2)\sim IH^{I}(\nu_1,\nu_2;\alpha,\beta,\gamma)$. Then, X_1/X_2 and X_1+X_2 are independent. Further, $X_1/X_2\sim B^{II}(\nu_1,\nu_2)$.*

4. Products of Two Independent Random Variables

Let (X_1,X_2) and X_3 be independent, $(X_1,X_2) \sim IH^I(\nu_1,\nu_2;\alpha,\beta,\gamma)$. In this section we derive density functions of (X_1X_3,X_2X_3) when

(i) $X_3 \sim IH^I(\kappa,\mu,\rho,\sigma)$,

(ii) $X_3 \sim B^{II}(\kappa,\sigma)$,

(iii) $X_3 \sim KB(\kappa,\mu,\lambda)$,

(iv) $X_3 \sim B^I(\kappa,\mu)$,

(v) $X_3 \sim B^{III}(\kappa,\mu)$, and

(vi) $X_3 \sim H^I(\kappa,\mu,\rho,\sigma)$.

Throughout this section we write $\nu_1+\nu_2=\nu$.

Theorem 3. *Let (X_1,X_2) and X_3 be independent, $(X_1,X_2) \sim IH^I(\nu_1,\nu_2;\alpha,\beta,\gamma)$, and $X_3 \sim IH^I(\kappa,\mu,\rho,\sigma)$. Then, the p.d.f. of $(Z_1,Z_2)=(X_1,X_2)X_3$ is given by*

$$\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)}\frac{\Gamma(\sigma+\kappa-\mu)\Gamma(\sigma+\kappa-\rho)}{\Gamma(\sigma)\Gamma(\kappa)\Gamma(\sigma+\kappa-\mu-\rho)}\frac{\Gamma(\nu+\sigma)\Gamma(\kappa+\gamma)}{\Gamma(\nu+\gamma+\kappa+\sigma)}$$
$$\times z_1^{\nu_1-1}z_2^{\nu_2-1}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_r(\mu)_s(\beta)_r(\rho)_s(\kappa+\gamma)_r(\nu+\sigma)_s}{(\gamma)_r(\sigma)_s(\nu+\kappa+\gamma+\sigma)_{r+s}\,r!\,s!}$$
$$\times {}_2F_1(\nu+\sigma+s,\nu+\gamma+r;\nu+\kappa+\gamma+\sigma+r+s;1-z_1-z_2), \tag{6}$$

where $z_1>0$ and $z_2>0$.

Proof. Using independence, the joint p.d.f. of (X_1,X_2) and X_3 is given by

$$\frac{K_1 x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\kappa-1}}{(1+x_1+x_2)^{\nu+\gamma}(1+x_3)^{\kappa+\sigma}}\,{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right){}_2F_1\left(\mu,\rho;\sigma;\frac{1}{1+x_3}\right),$$

where $x_1>0$, $x_2>0$, $x_3>0$ and

$$K_1=\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)}\frac{\Gamma(\sigma+\kappa-\mu)\Gamma(\sigma+\kappa-\rho)}{\Gamma(\sigma)\Gamma(\kappa)\Gamma(\sigma+\kappa-\mu-\rho)}.$$

Transforming $Z_1=X_1X_3$, $Z_2=X_2X_3$ $U=1/(1+X_3)$ with the Jacobian $J(x_1,x_2,x_3\to z_1,z_2,u)=1/(1-u)^2$ in the joint density of (X_1,X_2) and X_2 and integrating u, we obtain the p.d.f. of (Z_1,Z_2) as

$$K_1 z_1^{\nu_1-1}z_2^{\nu_2-1}\int_0^1\frac{u^{\nu+\sigma-1}(1-u)^{\kappa+\gamma-1}}{[1-(1-z_1-z_2)u]^{\nu+\gamma}}\,{}_2F_1\left(\alpha,\beta;\gamma;\frac{1-u}{1-(1-z_1-z_2)u}\right)$$
$$\times\,{}_2F_1(\mu,\rho;\sigma;u)\,\mathrm{d}u. \tag{7}$$

Now, expanding Gauss hypergeometric functions in the integral (7) in terms of power series we arrive at

$$K_1 z_1^{\nu_1-1} z_2^{\nu_2-1} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_r(\mu)_s(\beta)_r(\rho)_s}{(\gamma)_r(\sigma)_s r!s!} \int_0^1 \frac{u^{\nu+\sigma+s-1}(1-u)^{\kappa+\gamma+r-1}}{[1-(1-z_1-z_2)u]^{\nu+\gamma+r}}\,\mathrm{d}u.$$

Finally, using (A.5) and substituting for K_1 we obtain the desired result.

Corollary 2. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$, and $X_3 \sim B^{II}(\kappa, \sigma)$. Then, the p.d.f of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\begin{aligned} &\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma+\nu-\alpha-\beta)} \frac{\Gamma(\kappa+\sigma)}{\Gamma(\sigma)\Gamma(\kappa)} \frac{\Gamma(\nu+\sigma)\Gamma(\kappa+\gamma)}{\Gamma(\nu+\kappa+\gamma+\sigma)} z_1^{\nu_1-1} z_2^{\nu_2-1} \\ &\times \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\kappa+\gamma)_r}{(\gamma)_r(\nu+\kappa+\gamma+\sigma)_r\, r!} \\ &\times {}_2F_1(\nu+\sigma, \nu+\gamma+r; \nu+\kappa+\gamma+\sigma+r; 1-z_1-z_2), \end{aligned} \tag{8}$$

where $z_1 > 0$ and $z_2 > 0$.

Corollary 3. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim D^{II}(\nu_1, \nu_2; \gamma)$, and $X_3 \sim B^{II}(\kappa, \sigma)$. Then, the p.d.f of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\begin{aligned} &\frac{\Gamma(\gamma+\nu)}{\Gamma(\gamma)\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\kappa+\sigma)}{\Gamma(\sigma)\Gamma(\kappa)} \frac{\Gamma(\nu+\sigma)\Gamma(\kappa+\gamma)}{\Gamma(\nu+\kappa+\gamma+\sigma)} \\ &\times z_1^{\nu_1-1} z_2^{\nu_2-1} {}_2F_1(\nu+\sigma, \nu+\gamma; \nu+\kappa+\gamma+\sigma; 1-z_1-z_2), \end{aligned} \tag{9}$$

where $z_1 > 0$ and $z_2 > 0$.

Note that the Gauss hypergeometric functions in the densities (6), (8) and (9) can be expanded in series form if $0 < z_1 + z_2 < 1$. However, if $z_1 + z_2 > 1$, then $1 - 1/(z_1 + z_2) < 1$ and we use (A.6) to rewrite the densities (6), (8) and (9) in series involving Gauss hypergeometric functions having $1 - 1/(z_1 + z_2)$ as argument.

The next theorem gives the density of the product of Kummer-beta and inverted hypergeometric function type I variables.

Theorem 4. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$ and $X_3 \sim KB(\kappa, \mu, \lambda)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)X_3$ is*

$$\begin{aligned} &\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)\Gamma(\mu+\kappa)\Gamma(\gamma+\kappa)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)\Gamma(\gamma+\mu+\kappa)} \{{}_1F_1(\mu; \kappa+\mu; \lambda)\}^{-1} \\ &\times \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\gamma+\kappa)_r}{(\gamma+\mu+\kappa)_r\,(\gamma)_r\, r!} (1+z_1+z_2)^{-r} \\ &\times \mathbf{\Phi}_1\left[\mu, \nu+\gamma+r; \gamma+\mu+\kappa+r; \frac{1}{1+z_1+z_2}, \lambda\right], \quad z_1 > 0, \quad z_2 > 0. \end{aligned}$$

Proof. The joint p.d.f. of (X_1, X_2) and X_3 is given by

$$K_2 \frac{x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\kappa-1}(1-x_3)^{\mu-1}}{(1+x_1+x_2)^{\nu+\gamma}} {}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right)\exp[\lambda(1-x_3)], \tag{10}$$

where $x_1 > 0, x_2 > 0$, $0 < x_2 < 1$ and

$$K_2 = \frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)}\{B(\kappa,\mu)_1F_1(\mu;\kappa+\mu;\lambda)\}^{-1}.$$

Transforming $Z_1 = X_1 X_3$, $Z_2 = X_1 X_3$ and $W = 1 - X_3$ with the Jacobian $J(x_1, x_2, x_3 \to z_1, z_2, w) = 1/(1-w)^2$ in (10) and integrating w, we obtain the joint p.d.f. of Z_1 and Z_2 as

$$\begin{aligned} K_2 \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} &\int_0^1 \frac{w^{\mu-1}(1-w)^{\gamma+\kappa-1}}{\left[1-w/(1+z_1+z_2)\right]^{\nu+\gamma}} \\ &\times \exp(\lambda w)_2F_1\left(\alpha,\beta;\gamma;\frac{(1+z_1+z_2)^{-1}(1-w)}{1-w/(1+z_1+z_2)}\right)\mathrm{d}w. \end{aligned} \tag{11}$$

Now, expanding Gauss hypergeometric functions in the integral (11) in terms of power series we arrive at

$$K_2 \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!}(1+z_1+z_2)^{-r} \int_0^1 \frac{w^{\mu-1}(1-w)^{\gamma+\kappa+r-1}\exp(\lambda w)}{\left[1-w/(1+z_1+z_2)\right]^{\nu+\gamma+r}}\mathrm{d}w.$$

Finally, applying (A.12) and substituting for K_2 we obtain the desired result.

Corollary 4. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim D^{II}(\nu_1, \nu_2; \gamma)$ and $X_3 \sim KB(\kappa, \mu, \lambda)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\begin{aligned} &\frac{\Gamma(\gamma+\nu)\Gamma(\mu+\kappa)\Gamma(\gamma+\kappa)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\gamma+\mu+\kappa)}\{{}_1F_1(\mu;\kappa+\mu;\lambda)\}^{-1} \\ &\quad\times \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \mathbf{\Phi}_1\left[\mu,\nu+\gamma;\gamma+\mu+\kappa;\frac{1}{1+z_1+z_2},\lambda\right], \quad z_1 > 0, \quad z_2 > 0. \end{aligned}$$

Corollary 5. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim D^{II}(\nu_1, \nu_2; \gamma)$ and $X_3 \sim B^I(\kappa, \mu)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\begin{aligned} &\frac{\Gamma(\gamma+\nu)\Gamma(\mu+\kappa)\Gamma(\gamma+\kappa)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\gamma+\mu+\kappa)} \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \\ &\quad\times {}_2F_1\left(\mu,\nu+\gamma;\gamma+\mu+\kappa;\frac{1}{1+z_1+z_2}\right), \quad z_1 > 0, \quad z_2 > 0. \end{aligned}$$

Corollary 6. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$ and $X_3 \sim B^I(\kappa, \mu)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)\Gamma(\mu+\kappa)\Gamma(\gamma+\kappa)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)\Gamma(\gamma+\mu+\kappa)}$$
$$\times \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\gamma+\kappa)_r}{(\gamma+\mu+\kappa)_r(\gamma)_r\, r!}(1+z_1+z_2)^{-r}$$
$$\times {}_2F_1\left(\mu, \nu+\gamma+r; \gamma+\mu+\kappa+r; \frac{1}{1+z_1+z_2}\right), \quad z_1 > 0, \quad z_2 > 0.$$

Theorem 5. *Let (X_1, X_2) and X_3 be independent, $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$ and $X_3 \sim B^{III}(\kappa, \mu)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)X_3$ is given by*

$$\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)\Gamma(\kappa+\mu)\Gamma(\kappa+\gamma)}{2^{\mu}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)\Gamma(\kappa+\mu+\gamma)}$$
$$\times \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r(\gamma+\kappa)_r}{(\gamma)_r(\gamma+\kappa+\mu)_r\, r!}(1+z_1+z_2)^{-r}$$
$$\times F_1\left(\mu; \nu+\gamma+r, \mu+\kappa; \gamma+\kappa+\mu+r; \frac{1}{1+z_1+z_2}, \frac{1}{2}\right),$$

where $z_1 > 0$ and $z_2 > 0$.

Proof. The joint p.d.f. of (X_1, X_2) and X_3 is given by

$$K_3 \frac{x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\kappa-1}(1-x_3)^{\mu-1}}{(1+x_1+x_2)^{\nu+\gamma}(1+x_3)^{\kappa+\mu}} {}_2F_1\left(\alpha, \beta; \gamma; \frac{1}{1+x_1+x_2}\right), \tag{12}$$

where $x_1 > 0$, $x_2 > 0$, $0 < x_3 < 1$ and

$$K_3 = \frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)} 2^{\kappa}\{B(\kappa,\mu)\}^{-1}.$$

Now, transforming $Z_1 = X_1X_3$, $Z_2 = X_2X_3$ and $W = 1 - X_3$ with the Jacobian $J(x_1, x_2, x_3 \to z_1, z_2, w) = 1/(1-w)^2$ in (12) and integrating w, the marginal p.d.f. of (Z_1, Z_2) is derived as

$$\frac{K_3 z_1^{\nu_1-1} z_2^{\nu_2-1}}{2^{\kappa+\mu}(1+z_1+z_2)^{\nu+\gamma}} \int_0^1 \frac{w^{\mu-1}(1-w)^{\kappa+\gamma-1}}{\left[1-w/(1+z_1+z_2)\right]^{\nu+\gamma}(1-w/2)^{\kappa+\mu}}$$
$$\times {}_2F_1\left(\alpha, \beta; \gamma; \frac{(1+z_1+z_2)^{-1}(1-w)}{1-w/(1+z_1+z_2)}\right) \mathrm{d}w. \tag{13}$$

Expanding Gauss hypergeometric functions in the integral (13) in series form we arrive at

$$\frac{K_3 z_1^{\nu_1-1} z_2^{\nu_2-1}}{2^{\kappa+\mu}(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r\, r!(1+z_1+z_2)^r} \int_0^1 \frac{w^{\mu-1}(1-w)^{\kappa+\gamma+r-1}}{\left[1-w/(1+z_1+z_2)\right]^{\nu+\gamma+r}(1-w/2)^{\kappa+\mu}} \mathrm{d}w.$$

Finally, the desired result follows by using (A.11) and substituting for K_3.

Corollary 7. *Let* (X_1,X_2) *and* X_3 *be independent,* $(X_1,X_2)\sim D^{II}(\nu_1,\nu_2;\gamma)$ *and* $X_3\sim B^{III}(\kappa,\mu)$*. Then, the p.d.f. of* $(Z_1,Z_2)=(X_1,X_2)X_3$ *is given by*

$$\frac{\Gamma(\nu+\gamma)\Gamma(\kappa+\mu)\Gamma(\kappa+\gamma)}{2^{\mu}\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\kappa+\mu+\gamma)}\frac{z_1^{\nu_1-1}z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}}$$
$$\times F_1\left(\mu;\nu+\gamma,\mu+\kappa;\gamma+\kappa+\mu;\frac{1}{1+z_1+z_2},\frac{1}{2}\right),$$

where $z_1>0$ *and* $z_2>0$.

Theorem 6. *Let* (X_1,X_2) *and* X_3 *be independent,* $(X_1,X_2)\sim IH^I(\nu_1,\nu_2;\alpha,\beta,\gamma)$ *and* $X_3\sim H^I(\kappa,\mu,\rho,\sigma)$*. Then, the p.d.f. of* $(Z_1,Z_2)=(X_1,X_2)X_3$ *is given by*

$$\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)\Gamma(\kappa+\gamma)\Gamma(\sigma+\kappa-\mu)\Gamma(\sigma+\kappa-\rho)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)\Gamma(\sigma+\kappa-\mu-\rho)\Gamma(\kappa+\gamma+\sigma)}$$
$$\times\frac{z_1^{\nu_1-1}z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\mu)_s(\rho)_s(\alpha)_r(\beta)_r(\kappa+\gamma)_r}{(\gamma)_r(\kappa+\gamma+\sigma)_{s+r}\,s!\,r!}(1+z_1+z_2)^{-r}$$
$$\times{}_2F_1\left(\sigma+s,\nu+\gamma+r;\kappa+\sigma+\gamma+s+r;\frac{1}{1+z_1+z_2}\right),$$

where $z_1>0$ *and* $z_2>0$.

Proof. The joint p.d.f. of (X_1,X_2) and X_3 is given by

$$\frac{K_4x_1^{\nu_1-1}x_2^{\nu_2-1}x_3^{\kappa-1}(1-x_3)^{\sigma-1}}{(1+x_1+x_2)^{\nu+\gamma}}{}_2F_1\left(\alpha,\beta;\gamma;\frac{1}{1+x_1+x_2}\right){}_2F_1(\mu,\rho;\sigma;1-x_3),\tag{14}$$

where $x_1>0$, $x_2>0$, $0<x_3<1$ and

$$K_4=\frac{\Gamma(\nu+\gamma-\alpha)\Gamma(\nu+\gamma-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\nu+\gamma-\alpha-\beta)}\frac{\Gamma(\sigma+\kappa-\mu)\Gamma(\sigma+\kappa-\rho)}{\Gamma(\sigma)\Gamma(\kappa)\Gamma(\sigma+\kappa-\mu-\rho)}.$$

Now, transforming $Z_1=X_1X_3$, $Z_2=X_2X_3$ and $W=1-X_3$ with the Jacobian $J(x_1,x_2,x_3\rightarrow z_1,z_2,w)=1/(1-w)^2$ in (14) and integrating w, we obtain the p.d.f. of (Z_1,Z_2) as

$$\frac{K_4z_1^{\nu_1-1}z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}}\int_0^1\frac{w^{\sigma-1}(1-w)^{\kappa+\gamma-1}}{\left[1-w/(1+z_1+z_2)\right]^{\nu+\gamma}}$$
$$\times{}_2F_1\left(\alpha,\beta;\gamma;\frac{(1+z_1+z_2)^{-1}(1-w)}{1-w/(1+z_1+z_2)}\right){}_2F_1\left(\mu,\rho;\sigma;w\right)\,\mathrm{d}w,\quad z_1>0,\quad z_2>0.$$

Now, expanding the Gauss hypergeometric functions in series form, integrating the resulting expression using (A.5), substituting for K_4 and simplifying, we obtain the desired result.

Corollary 8. *Let* (X_1, X_2) *and* X_3 *be independent,* $(X_1, X_2) \sim D^{II}(\nu_1, \nu_2; \gamma)$ *and* $X_3 \sim H^I(\kappa, \mu, \rho, \sigma)$. *Then, the p.d.f. of* $(Z_1, Z_2) = (X_1, X_2)X_3$ *is given by*

$$\frac{\Gamma(\nu+\gamma)\Gamma(\kappa+\gamma)\Gamma(\sigma+\kappa-\mu)\Gamma(\sigma+\kappa-\rho)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\kappa)\Gamma(\gamma)\Gamma(\sigma+\kappa-\mu-\rho)\Gamma(\kappa+\gamma+\sigma)} \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \times \sum_{s=0}^{\infty} \frac{(\mu)_s (\rho)_s}{(\kappa+\gamma+\sigma)_s \, s!} {}_2F_1\left(\sigma+s, \nu+\gamma; \kappa+\sigma+\gamma+s; \frac{1}{1+z_1+z_2}\right),$$

where $z_1 > 0$ *and* $z_2 > 0$.

Theorem 7. *Let* (X_1, X_2), Y_1 *and* Y_2 *be independent,* $(X_1, X_2) \sim IH^I(\nu_1, \nu_2; \alpha, \beta, \gamma)$ *and* $Y_i \sim B^I(a_i, b_i)$, $i = 1, 2$. *Then, the p.d.f of* $(Z_1, Z_2) = (X_1, X_2)Y_1Y_2$ *is given by*

$$\frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(a_1+\gamma)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_1+b_1+b_2+\gamma)} \frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(\gamma+\nu-\alpha-\beta)} \times \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(b_2)_s (a_1+b_1-a_2)_s (\alpha)_r (\beta)_r (a_1+\gamma)_r}{(\gamma)_r (a_1+b_1+b_2+\gamma)_{s+r} \, s! \, r!} (1+z_1+z_2)^{-r} \times {}_2F_1\left(b_1+b_2+s, \nu+\gamma+r; a_1+b_1+b_2+\gamma+s+r; \frac{1}{1+z_1+z_2}\right),$$

where $z_1 > 0$ *and* $z_2 > 0$.

Proof. Using Theorem A.8, $Y_1Y_2 \sim H^I(a_1, b_2, a_1+b_1-a_2, b_1+b_2)$. Now, using independence of (X_1, X_2) and X_3 and Theorem 6, we obtain the desired result.

Corollary 9. *Let* (X_1, X_2), Y_1 *and* Y_2 *be independent,* $(X_1, X_2) \sim D^{II}(\nu_1, \nu_2; \gamma)$ *and* $Y_i \sim B^I(a_i, b_i)$, $i = 1, 2$. *Then, the p.d.f of* $(Z_1, Z_2) = (X_1, X_2)Y_1Y_2$ *is given by*

$$\frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(a_1+\gamma)\Gamma(\nu+\gamma)}{\Gamma(a_1)\Gamma(a_2)\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\gamma)\Gamma(a_1+b_1+b_2+\gamma)} \frac{z_1^{\nu_1-1} z_2^{\nu_2-1}}{(1+z_1+z_2)^{\nu+\gamma}} \times \sum_{s=0}^{\infty} \frac{(b_2)_s (a_1+b_1-a_2)_s}{(a_1+b_1+b_2+\gamma)_s \, s!} {}_2F_1\left(b_1+b_2+s, \nu+\gamma; a_1+b_1+b_2+\gamma+s; \frac{1}{1+z_1+z_2}\right),$$

where $z_1 > 0$ *and* $z_2 > 0$.

Appendix: Some Known Definitions and Results

Here, we give some definitions and additional results which are used throughout this work. We use the Pochhammer symbol $(a)_n$ defined by

$$(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1) \text{ for } n = 1, 2, \ldots, \ (a)_0 = 1.$$

The generalized hypergeometric function of scalar argument is defined by

$$
{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{A.1}
$$

where a_i, $i=1,\dots,p$; b_j, $j=1,\dots,q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (A.1) are available in the literature, see Luke [5]. From (A.1) it is easy to see that

$$
{}_0F_0(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}=\exp(z),\quad {}_1F_1(a;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k!},
$$

and

$$
{}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},\quad |z|<1.
$$

Also, under suitable conditions, we have from Luke [5, Eq. 3.6(10)],

$$
\begin{aligned}
&\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;zy)\,\mathrm{d}z\\
&\quad=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}{}_{p+1}F_{q+1}(a_1,\dots,a_p,\alpha;b_1,\dots,b_q,\alpha+\beta;y)
\end{aligned} \tag{A.2}
$$

and Luke [5, Eq. 3.6(13)],

$$
\begin{aligned}
&\int_0^\infty \exp(-\delta z)z^{\alpha-1}{}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;zy)\,\mathrm{d}z\\
&\quad=\Gamma(\alpha)\delta^{-\alpha}{}_{p+1}F_q(a_1,\dots,a_p,\alpha;b_1,\dots,b_q;\delta^{-1}y).
\end{aligned} \tag{A.3}
$$

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$
{}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,\mathrm{d}t, \tag{A.4}
$$

and

$$
{}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t, \tag{A.5}
$$

respectively, where $\mathrm{Re}(a)>0$ and $\mathrm{Re}(c-a)>0$. It is easy to check by using (A.5) that

$$
\begin{aligned}
{}_2F_1(a,b;c;z) &= (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{-z}{1-z}\right)\\
&= (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{-z}{1-z}\right)
\end{aligned} \tag{A.6}
$$

and for $\text{Re}(a) > 0$, $\text{Re}(b+1-c) > 0$ and $|\arg(z)| < \pi$, it has been shown that [see Luke 5, Eq. 3.6.3],

$$ {}_2F_1(a,b;a+b+1-c;1-z) = \frac{\Gamma(a+b+1-c)}{\Gamma(b)\Gamma(a+1-c)} \int_0^\infty \frac{s^{b-1}(1+s)^{c-b-1}\,\mathrm{d}s}{(1+sz)^a}. \tag{A.7} $$

Further, for $\text{Re}(\lambda) < \text{Re}(\nu)$, we have [Prudnikov 12, Eq. 1.2.4.4],

$$ \int_x^\infty \frac{y^{\lambda-1}}{(y+a)^\nu}\mathrm{d}y = \frac{x^{\lambda-\nu}}{\nu-\lambda}\, {}_2F_1\left(\nu,\nu-\lambda;1+\nu-\lambda;-\frac{a}{x}\right). \tag{A.8} $$

The Appell's first hypergeometric function F_1 is defined by

$$ \begin{aligned} F_1(a;b_1,b_2;c;z_1,z_2) &= \sum_{r,s=0}^\infty \frac{(a)_{r+s}(b_1)_r(b_2)_s}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\,s!} \\ &= \sum_{r=0}^\infty \frac{(a)_r(b_1)_r}{(c)_r} \frac{z_1^r}{r!}\, {}_2F_1(a+r,b_2;c+r;z_2) \\ &= \sum_{s=0}^\infty \frac{(a)_s(b_2)_s}{(c)_s} \frac{z_2^s}{s!}\, {}_2F_1(a+s,b_1;c+s;z_1), \end{aligned} \tag{A.9} $$

where $|z_1| < 1$ and $|z_2| < 1$. The Humbert's confluent hypergeometric function $\mathbf{\Phi}_1$ is defined by

$$ \begin{aligned} \mathbf{\Phi}_1[a,b_1;c;z_1,z_2] &= \sum_{r,s=0}^\infty \frac{(a)_{r+s}(b_1)_r}{(c)_{r+s}} \frac{z_1^r z_2^s}{r!\,s!}, \\ &= \sum_{r=0}^\infty \frac{(a)_r(b_1)_r}{(c)_r} \frac{z_1^r}{r!}\, {}_1F_1(a+r;c+r;z_2) \\ &= \sum_{s=0}^\infty \frac{(a)_s}{(c)_s} \frac{z_2^s}{s!}\, {}_2F_1(a+s,b_1;c+s;z_1), \end{aligned} \tag{A.10} $$

where $|z_1| < 1$, $|z_2| < \infty$. The integral representations of F_1 and $\mathbf{\Phi}_1$ are given by

$$ F_1(a;b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1}\,\mathrm{d}v}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}}, \tag{A.11} $$

and

$$ \mathbf{\Phi}_1[a,b_1;c;z_1,z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1}\exp(vz_2)\,\mathrm{d}v}{(1-vz_1)^{b_1}}, \tag{A.12} $$

where $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Note that for $b_1 = 0$, F_1 and $\mathbf{\Phi}_1$ reduce to ${}_2F_1$ and ${}_1F_1$ functions, respectively. For properties and further results on these functions the reader is referred to Luke [5] and Srivastava and Karlsson [15].

Next, we define the beta type I, beta type II, beta type III, hypergeometric function type I and Kummer-beta distributions. These definitions can be found in Gordy [1], Johnson, Kotz and Balakrishnan [4], Nagar and Zarrazola [11], and Sánchez and Nagar [13].

Definition A.1. *The random variable X is said to have a beta type I distribution with parameters (a,b), $a>0$, $b>0$, denoted as $X\sim B^I(a,b)$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1},\quad 0<x<1,$$

where $B(a,b)$ is the beta function given by

$$B(a,b)=\Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}.$$

Definition A.2. *The random variable X is said to have a beta type II distribution with parameters (a,b), denoted as $X\sim B^{II}(a,b)$, $a>0$, $b>0$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1}x^{a-1}(1+x)^{-(a+b)},\quad x>0.$$

Definition A.3. *The random variable X is said to have a beta type III distribution with parameters (a,b), denoted as $X\sim B^{III}(a,b)$, $a>0$, $b>0$, if its p.d.f. is given by*

$$2^a\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1}(1+x)^{-(a+b)},\quad 0<x<1.$$

Definition A.4. *The random variable X is said to have a Kummer-beta distribution, denoted by $X\sim KB(\alpha,\beta,\lambda)$, if its p.d.f. is given by*

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}\exp\left[\lambda(1-x)\right]}{B(\alpha,\beta)_1F_1(\beta;\alpha+\beta;\lambda)},\quad 0<x<1,$$

where $\alpha>0$, $\beta>0$ and $-\infty<\lambda<\infty$.

Note that for $\lambda=0$ the above density simplifies to a beta type I density with parameters α and β.

The bivariate generalizations of beta type I and beta type II distributions are defined next.

Definition A.5. *The random variables X and Y are said to have a Dirichlet type I distribution of order 3 with parameters (a,b,c), $a>0$, $b>0$, $c>0$, denoted as $X\sim D^I(a,b;c)$, if their joint p.d.f. is given by*

$$\{B(a,b,c)\}^{-1}x^{a-1}y^{b-1}(1-x-y)^{c-1},\quad x>0,\quad y>0,\quad x+y<1,$$

where $B(a,b,c)$ is defined by

$$B(a,b,c)=\Gamma(a)\Gamma(b)\Gamma(c)\{\Gamma(a+b+c)\}^{-1}.$$

Definition A.6. *The random variables X and Y are said tohave a Dirichlet type II distribution of order 3 with parameters (a,b,c), $a>0$, $b>0$, $c>0$, denoted as $X\sim D^{II}(a,b;c)$, if their joint p.d.f. is given by*

$$\{B(a,b,c)\}^{-1}x^{a-1}y^{b-1}(1+x+y)^{-(a+b+c)},\quad x>0,\quad y>0.$$

Definition A.7. *The random variable X is said to have a hypergeometric function type I distribution, denoted by $X \sim H^I(\nu, \alpha, \beta, \gamma)$, if its p.d.f. is given by*

$$\frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)} x^{\nu-1}(1-x)^{\gamma-1} {}_2F_1(\alpha,\beta;\gamma;1-x), \quad 0 < x < 1,$$

where $\gamma+\nu-\alpha-\beta > 0$, $\gamma > 0$ and $\nu > 0$.

The following result (Gupta and Nagar [3], Nagar and Alvarez [8]) states that the hypergeometric function type I distribution can be obtained as the distribution of the product of two independent beta type I variables.

Theorem A.8. *Let X_1 and X_2 be independent, $X_i \sim B^I(a_i, b_i)$, $i = 1, 2$. Then, $X_1X_2 \sim H^I(a_1, b_2, a_1 + b_1 - a_2, b_1 + b_2)$.*

The matrix variate generalizations of beta type I, beta type II, beta type III, hypergeometric function type I and Kummer-beta distributions have been defined and studied extensively. For example, see Gupta and Nagar [2], Gupta and Nagar [3], and Nagar and Gupta [10].

Acknowledgments The research work of DKN was supported by the Comité para el Desarrollo de la Investigación, Universidad de Antioquia research grant no. IN560CE.

References

[1] M. B. Gordy. Computationally convenient distributional assumptions for common-value auctions. *Comput. Econom.*, 12:61–78, 1998.

[2] A. K. Gupta and D. K. Nagar. Matrix variate beta distribution. *Int. J. Math. Math. Sci.*, 24(7):449–459, 2000.

[3] A. K. Gupta and D. K. Nagar. *Matrix Variate Distributions*, volume 104 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, 2000.

[4] N. L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous Univariate Distributions. Vol. 2*. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, 1995.

[5] Y. L. Luke. *The Special Functions and Their Approximations. Vol. I*, volume 53 of *Mathematics in Science and Engineering*. Academic Press, New York, 1969.

[6] A. W. Marshall and I. Olkin. *Inequalities: theory of majorization and its applications*, volume 143 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[7] S. Nadarajah. Reliability for some bivariate beta distributions. *Math. Probl. Eng.*, (1):101–111, 2005.

[8] D. K. Nagar and J. A. Alvarez. Properties of the hypergeometric function type I distribution. *Adv. Appl. Stat.*, 5(3):341–351, 2005.

[9] D. K. Nagar, P. A. Bran-Cardona, and A. K. Gupta. Multivariate generalization of the hypergeometric function type I distribution. *Acta Appl. Math.*, 105(1):111–122, 2009.

[10] D. K. Nagar and A. K. Gupta. Matrix-variate Kummer-beta distribution. *J. Aust. Math. Soc.*, 73(1):11–25, 2002.

[11] D. K. Nagar and E. Zarrazola. Distributions of the product and the quotient of independent Kummer-beta variables. *Sci. Math. Jpn.*, 61(1):109–117, 2005.

[12] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and series. Vol. 1*. Gordon & Breach Science Publishers, New York, 1986. Elementary functions, Translated from the Russian and with a preface by N. M. Queen.

[13] L. E. Sánchez and D. K. Nagar. Distributions of the product and the quotient of independent beta type 3 variables. *Far East J. Theor. Stat.*, 17(2):239–251, 2005.

[14] B. D. Sivazlian. On a multivariate extension of the gamma and beta distributions. *SIAM J. Appl. Math.*, 41(2):205–209, 1981.

[15] H. M. Srivastava and P. W. Karlsson. *Multiple Gaussian Hypergeometric Series*. Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester, 1985.

[16] E. Zarrazola and D. K. Nagar. Product of independent inverted hypergeometric function type I variables. *Revista Ingeniería y Ciencia, Universidad Eafit*, 5(10):93–106, 2009.