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Sequentially pure monomorphisms of acts over semigroups

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Abstract. Any notion of purity is normally defined in terms of solvability of some set of equations. In this paper we first take this point of view to introduce a kind of purity, called sequential purity, for acts over semigroups (which is of some interest to computer scientists, too), and then show that it is actually equivalent to C^p-purity resulting from a closure operator.
The main objective of the paper is to study properties of the category of all acts over a semigroup with respect to sequentially pure monomorphisms. These properties are usually needed to study the homological notions, such as injectivity, of acts.

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1. Introduction and Preliminaries

To study mathematical notions in a category $\mathscr{A}$ with respect to a class $\mathscr{M}$ of its morphisms, one should know some of the categorical properties of the pair $(\mathscr{A}, \mathscr{M})$.

One of the very useful categories in many branches of mathematics, as well as in computer sciences, is the category **Act-S** of sets with a right action of a semigroup S on them. In this paper we take $\mathscr{A}$ to be this category and $\mathscr{M}_p$ to be a particularly interesting class of monomorphisms, to be called *sequentially pure*, and investigate its categorical properties. First we give the following preliminaries needed in the sequel.

1.1. The category of acts over semigroups

First recall the following, for example from [13] or [5]. Let S be a semigroup and A be a set. If we have a mapping

$$\begin{aligned} \mu: \quad & A \times S \to A \\ & (a,s) \longmapsto as := \mu(a,s) \end{aligned}$$

such that $a(st) = (as)t$ for $a \in A,\ s,t \in S$, we call A a (*right*) S-act or a (*right*) act over S.

If S is a monoid with an identity 1, we usually also require that $a1 = a$ for $a \in A$.

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In fact, an S-act is a universal algebra $(A, (\mu_s)_{s\in S})$ where each $\mu_s : A \to A$ is a unary operation on A such that $\mu_s \circ \mu_t = \mu_{st}$ for each $s, t \in S$, and $\mu_1 = id_A$ if S has an identity 1.

An element a of an S-act A is called a *fixed* or a *zero* element if $as = a$ for all $s \in S$. Note that one can always adjoin a zero element to A and get an act $A^0 = A \cup \{0\}$ with a zero element.

Let A be an S-act and $A' \subseteq A$. Then A' is called a *subact* of A if $a's \in A'$ for all $s \in S$ and $a' \in A'$. Note that the semigroup S can itself be regarded as an S-act with its multiplication as its action. A subact of the S-act S is called a *right ideal* of the semigroup S.

A *homomorphism* (also called an *equivariant* map, or an *S-map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A$ and $s \in S$, $f(as) = f(a)s$.

Since id_A and the composition of two equivariant maps are equivariant, we have the category **Act-S** of all (right) S-sets and S-maps between them.

Recall that to every semigroup S without an identity one can adjoin an identity 1 by setting $1s = s = s1$ for all $s \in S$ and get an S-act denoted by S^1.

As a very interesting example, used in computer sciences as a convenient means of algebraic specification of process algebras (see [7, 8]), consider the monoid $(\mathbb{N}^\infty, \cdot)$, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $n < \infty, \forall n \in \mathbb{N}$ and $m \cdot n = min\{m, n\}$ for $m, n \in \mathbb{N}^\infty$. Then an $\mathbb{N}^\infty$-act is called a *projection algebra* or a *projection space* (see also [11, 14]).

1.2. Some ingredients of the category Act-S

In this subsection we give some categorical and algebraic ingredients of **Act-S** needed in the sequel.

Since the class of S-acts is an equational class, the category **Act-S** is complete (has all products and equalizers). In fact, limits in this category are computed as in the category **Set** of sets and equipped with a natural action. In particular, the terminal object of **Act-S** is the singleton $\{0\}$, with the obvious S-action. Also, for S-acts A, B, their cartesian product $A \times B$ with the S-action defined by $(a, b)s = (as, bs)$ is the product of A and B in **Act-S**.

Recall that for a family $\{A_\alpha : \alpha \in I\}$ of S-acts each with a unique fixed element 0, the *direct sum* $\bigoplus_{\alpha\in I} A_\alpha$ is defined to be the subact of the product $\prod_{\alpha\in I} A_\alpha$ consisting of all $(a_\alpha)_{\alpha\in I}$ such that $a_\alpha = 0$ for all $\alpha \in I$ except a finite number of indices.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

in **Act-S** is the subact $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of projection maps. Notice that for the case where g is a monomorphism, P can be taken as (isomorphic to) $f^{-1}(C)$.

All colimits in **Act-S** exist and are calculated as in **Set** with a natural action of S on them. In particular, $\emptyset$ with the empty action of S on it is the initial object of **Act-S**. Also, the coproduct of two S-acts A, B is their disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ with the action of S on

$A \sqcup B$ defined by $(a,1)s = (as,1)$, $(b,2)s = (bs,2)$ for $s \in S$, $a \in A$, $b \in B$. The coproduct $A \sqcup \{0\}$ is denoted by A^0.

Definition 1.1. Let A be an S-act. An equivalence relation ρ on A is called an S-*act congruence* on A if $a\rho a'$ implies $as\rho a's$ for $a, a' \in A$, $s \in S$. The right action making A/ρ an S-act is defined by $[a]s = [as]$.

For $H \subseteq A \times A$, the *congruence generated by* H, that is the smallest congruence on A containing H, is denoted by $\rho(H)$.

For a subset H of $A \times A$ let H^e be the equivalence relation generated by H and

$$H^c = \{(as, bs) : (a,b) \in H, s \in S\}$$

Then we have:

Lemma 1.1. *Let A be an act over a semigroup S and $H \subseteq A \times A$. Then $\rho(H) = (H \cup H^c)^e$.*

Proof. It is enough to show that the equivalence relation $(H \cup H^c)^e$ is a congruence. Suppose that $x(H \cup H^c)^e y$ and $s \in S$. Then $x = y$, and so $xs = ys$, or there exist $z_1, \ldots, z_n \in A$ such that $x = z_1, y = z_n$ and $(z_i, z_{i+1}) \in (H \cup H^c) \cup (H \cup H^c)^{-1}$. Then we also have $(z_i s, z_{i+1} s) \in (H \cup H^c) \cup (H \cup H^c)^{-1}$.

Corollary 1.1. *Let $H \subseteq A \times A$ and $\rho = \rho(H)$. Then, for $a, b \in A$, one has $a\rho b$ if and only if either $a = b$ or there exist $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n \in A, s_1, s_2, \ldots, s_n \in S^1$ where for $i = 1, \ldots, n, (p_i, q_i) \in H \cup H^{-1}$, such that $a = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, \ldots, q_n s_n = b$.*

The pushout of a given diagram

$$\begin{array}{ccc} & A & \xrightarrow{f} & B \\ g & \downarrow & & \\ & C & & \end{array}$$

in **Act-S** is the factor act $Q = (B \sqcup C)/\theta$ where $\theta = \rho(H)$ and H consists of all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C$, $u_C : C \to B \sqcup C$ are coproduct injections. Also, the pushout maps are given as $q_1 = \pi u_C : C \to (B \sqcup C)/\theta$, $q_2 = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Act-S** are constructed analogously.

Recall that a directed system of S-acts and S-maps is a family $(B_\alpha)_{\alpha \in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha \leq \beta \in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, also $g_{\alpha\alpha} = id$. Note that the *direct limit* (directed colimit) of a directed system $((B_\alpha)_{\alpha \in I}, (g_{\alpha\beta})_{\alpha \leq \beta \in I})$ in **Act-S** is given as $\underrightarrow{lim}_\alpha B_\alpha = \coprod_\alpha B_\alpha / \rho$ where the congruence ρ is given by $b_\alpha \rho b_\beta$ if and only if there exists $\gamma \geq \alpha, \beta$ such that $u_\gamma g_{\alpha\gamma}(b_\alpha) = u_\gamma g_{\beta\gamma}(b_\beta)$, in which each $u_\alpha : B_\alpha \to \coprod_\alpha B_\alpha$ is an injection map of the coproduct. Notice that the family $g_\alpha = \pi u_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \leq \beta$, where $\pi : \coprod_\alpha B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ is the natural S-map.

The left and the right adjoints F and H, respectively, of the forgetful functor U: **Act-S** $\to$ **Set** exist and are defined as follows:

The *free functor* F:**Set** $\to$ **Act-S** is defined by: $F(X) = X \times S^1$ with the S-action given by $(x,t)s = (x,ts)$, for $t \in S^1$, $s \in S$, $x \in X$, and for every map $f : X \to Y$ in **Set**, $F(f) = f \times id : X \times S^1 \to Y \times S^1$.

The existence of free S-acts, in particular on the singleton set, shows that an S-map is a monomorphism if and only if it is one-one. Therefore, we do not distinguish between monomorphisms of acts and inclusions.

The *cofree functor* H : **Set** $\to$ **Act-S** is defined by $HX = X^{S^1}$, the set of all *functions* from S^1 to the set X, with the action of S on X^{S^1} given by $(fs)(t) = f(st)$ for $f \in X^{S^1}, s \in S$, and $t \in S^1$. Also, for a function $h : X \to Y$, $H(h) : X^{S^1} \to Y^{S^1}$ is defined by $(Hh)(f) = hf$ for $f \in X^{S^1}$.

Since a left adjoint preserves colimits, the functor U preserves epimorphisms. So, epimorphisms in **Act-S** are exactly onto S-maps.

2. Sequentially Pure Monomorphisms

Any notion of pure monomorphisms is normally defined in terms of solvability of some set of equations. In the following we first consider this point of view to define a kind of pure monomorphisms, which is also of interest to computer scientists, which we are going to study their behaviour in this paper, and then show that they are actually equivalent to C^p*-pure monomorphisms* resulting from a closure operator on the category **Act-S**.

2.1. Sequentially pure monomorphisms

In [10, 14, 16], it is shown that the equations with constants from an S-act A are of one the following three types:

$$xs = yt, \quad xs = xt, \quad xs = a$$

where $s, t \in S, a \in A$. Here we are concerned with the equations of the type $xs = a$ only.

Gould in [10] defines an *α-system* of equations on an S-act A to be

$$\Sigma = \{xs_j = a_j : j \in J, |J| < \alpha, s_j \in S, a_j \in A\}$$

in which $s_i = s_j$ need not imply $a_i = a_j$. But, note that if for any $s \in S$ there exist two equations of the form $xs = a_1, xs = a_2$ in Σ and Σ has a solution b in some extension B of A, then $a_1 = a_2$. So, for a system Σ of equations to have a solution there can only be at most one equation $xs = a$ in Σ for each $s \in S$. Therefore, Σ should actually be taken of the form $\Sigma_T = \{xs = a_t : t \in T, a_t \in A\}$ for some $T \subseteq S$. Hence, for any fixed $T \subseteq S$, there is a one to one correspondence between the set of all systems of equations of the above form on an S-act A and the set of all functions $k : T \to A$. In fact, to each system of equations Σ_T we get the function $k_\Sigma : T \to A$ given by $k(t) = a_t$ and conversely, for any function $k : T \to A$ one has the system of equations $\Sigma_k = \{xt = k(t) \mid t \in T\}$ on A. Thus we have the following definition.

Definition 2.1. (1) Let T be a subset of S and A be an S-act. Any $\Sigma_T = \{xs = a_t : t \in T, a_t \in A\}$ (or, equivalently, a map $k : T \to A$) will be called a *T-system of equations* (or a *T-sequence*) on A.

(2) We say that a system Σ_T (or $k : T \to A$) is *solvable* in an extension B of A if there is some $b \in B$ such that $bt = a_t$ (or $k(t) = bt$) for all $t \in T$; that is, $k = \lambda_b$, where $\lambda_b(t) = bt$.

(3) We say that a system Σ_T (or $k : T \to A$) is *consistent* if it has a solution in some extension B of A.

Now, we are going to show that we should actually only consider just I-sequences for an *ideal* I of S rather than any T-sequence for any *subset* T of S. Note that, although an I-sequence $k : I \to A$ is just a *function* and not necessarily a *homomorphism*, we have the following:

Theorem 2.1. *Let I be an ideal of S and A be an S-act. Then the following are equivalent for an I-sequence $k : I \to A$:*

(1) *$k : I \to A$ is a homomorphism.*

(2) *$k : I \to A$ is a consistent map.*

(3) *The system $\Sigma = \{xs = k(s) |\ s \in I\}$ is a consistent system.*

Proof. (1)⇒(2) Let $k : I \to A$ be a homomorphism. Let E be an injective S-act containing A (see [13]). Considering the extension I^1 of I, k can be lifted to $\bar{k} : I^1 \to E$. Now it is easily seen that $k = \lambda_x$ where $x = k(1) \in E$, and so k is consistent.

(2)⇒(1) Let k be a consistent map. So, there exist an extension B of A and $b \in B$ such that $k = \lambda_b$ which is in fact a homomorphism.

The equivalence of (3) and (2) follows just from the definition.

Corollary 2.1. *A T-system $\Sigma_T = \{xt = a_t |\ t \in T\}$ (or a T-sequence $k : T \to A$) is consistent if and only if $\hat{k} : TS^1 \to A$ defined by $\hat{k}(ts) = a_t s$ for $t \in T, s \in S^1$ is a "well defined" equivariant map.*

Proof. Clearly $\Sigma_T = \{xt = a_t |\ t \in T\}$ is consistent if and only if $\Sigma_1 = \{x(ts) = a_t s |\ t \in T, s \in S^1\}$ is consistent. Now, since TS^1 is a right ideal of S, the latter is true by the above theorem if and only if $\hat{k} : TS^1 \to A$ with $\hat{k}(ts) = a_t s$ is a homomorphism.

The above corollary shows that, as long as consistent systems of equations are concerned, we may as well consider only *equivariant maps* $k : I \to A$ from a right *ideal* I of S to A rather than *functions* from an arbitrary *set* T to A. In particular, we take I to be S itself and have the following:

Definition 2.2. Let A be a subact of B. We say that A is *sequentially pure* or *s-pure* in B if any one of the following equivalent conditions hold:

(1) Every $\Sigma = \{xs = a_s : s \in S, a_s \in A\}$ is solvable in A whenever it is solvable in B.

(2) For every $b \in B$ with $bS \subseteq A$ there is an element $a \in A$ such that $\lambda_b = \lambda_a$; in the sense that $bs = as$ for each $s \in S$.

(3) Every homomorphism $k : S \to A$ is of the form λ_a for some $a \in A$ whenever it is of the form λ_b for some $b \in B$.

A monomorphism $f : A \to B$ is said to be *s-pure* if $f(A)$ is s-pure in B.

The following is a simple fact which will be used later:

Lemma 2.1. *For every S-acts A and B, we have:*
(1) *A is s-pure in $A \sqcup B$.*
(2) *A is s-pure in A^0.*

Remark 2.1. Note that one may say that a subact A of an act B is *finitely pure* in B if for every finite subset $T \subseteq S$, the system Σ_T with constants from A has a solution in A whenever it has a solution in B. Clearly, if S is finitely generated, say by T, then by Corollary 2.1, finite (or even T-) purity implies s-purity. But, the converse is not true: Consider the semigroup $S = \{1, a, b\}$ in which 1 is a left identity and a, b are zero elements. Then $A = \{a, b\}$ is an s-pure subact of S, but the finite system $\Sigma = \{xa = a, xb = b\}$ having solution 1 in S does not have any solution in A.

2.2. Sequential purity versus C^p-purity

In this subsection we introduce a closure operator which is closely related to sequential purity defined above. We are not going to fully investigate the properties of this closure operator as in [4]. First recall the following definition of a categorical closure operator from [3]. Denoting the lattice of all subacts of an S-act B by $Sub(B)$, we have:

Definition 2.3. A family $C = (C_B)_{B\in\mathbf{Act-S}}$, with $C_B : Sub(B) \to Sub(B)$, taking any subact $A \leq B$ to a subact $C_B(A)$ (or $C(A)$, if no confusion arises) is called a *closure operator* on **Act-S** if it satisfies the following:
(c_1) (*Extension*) $A \leq C(A)$,
(c_2) (*Monotonicity*) $A_1 \leq A_2 \leq B$ implies $C(A_1) \leq C(A_2)$,
(c_3) (*Continuity*) $f(C_B(A)) \leq C_C(f(A))$ for all morphisms $f : B \to C$.

Now, one has the usual two classes of monomorphisms related to any closure operator as follows:

Definition 2.4. Let $A \leq B$ be in **Act-S**. We say that A is C-*closed* in B if $C(A) = A$, and it is C-*dense* in B if $C(A) = B$. Also, an S-map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) subact of B.

Now we recall the following closure operator needed in the sequel and has been studied in [4] and used in [6, 9, 14] to study a kind of injectivity.

Definition 2.5. For any subact A of an S-act B, define a closure operatorC^d by

$$C^d(A) = \{b \in B : bS \subseteq A\}$$

Now, note that A is C^d-dense (or simply s-dense) in an extension B of A if $C^d(A) = B$, that is, for every $b \in B$, $bS \subseteq A$.

Notice that in the case where S is a monoid, $C^d(A) = A$ for every $A \leq B$. So, it is more interesting to consider the closure operator C^d only for semigroups, or for semigroup part S of monoids of the form $T = S^1$.

We now introduce and study another closure operator on **Act-S** which will be shown to be closely related to sequential purity.

Definition 2.6. The *sequential pure closure operator* C^p on **Act-S** is defined as

$$C^p(A) = \{b \in B : \exists a \in A, \lambda_b = \lambda_a\}$$

where $\lambda_x : S \to A$ is defined by $\lambda_x(s) = xs$.

Now, note that A is C^p-dense in an extension B of A if $C^p(A) = B$ (this means that for every $b \in B$ there is an $a \in A$ with $\lambda_b = \lambda_a$; that is, $bs = as$ for every $s \in S$). And A is C^p-closed in B if $C^p(A) = A$ (that is, for every $b \in B - A$ and $a \in A$ there is an $s \in S$ with $bs \neq as$).

Notice that C^p-closedness is preserved by inverse image of S-maps and C^p-denseness is preserved by images of onto S-maps.

Some easily proved properties of this last closure operator is stated in the following:

Lemma 2.2. C^p *is:* (1) *a closure operator,* (2) *idempotent,* (3) *hereditary; for* $C \leq A \leq B$, $C^p_B(A) = C^p_B(C) \cap A$, (4) *weakly hereditary; every* $A \leq B$ *is* C^p*-dense in* $C^p_B(A)$, (5) *grounded;* $C^p(\emptyset) = \emptyset$, (6) *additive;* $C^p(A \cup C) = C^p(A) \cup C^p(C)$, (7) *fully additive;* $C^p(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} C^p(A_i)$, (8) $C^p_A(\bigcap A_i) \subseteq \bigcap C^p_A(A_i)$, (9) *productive; for every family of subacts* A_i *of* B_i, *taking* $A = \prod_i A_i$ *and* $B = \prod_i B_i$, $C^p_B(A) = \prod_i C^p_{B_i}(A_i)$.

And, some of the properties that C^p does not satisfy in general are:

Lemma 2.3. *For any semigroup* S, C^p *is not:* (1) *discrete;* $C^p_B(A) = A$ *for every* S*-act* B *and every* $A \leq A$, (2) *trivial;* $C^p_B(A) = B$ *for every* B *and every* $A \leq B$, (3) *minimal; for* $C \leq A \leq B$, $C^p_B(A) = A \cup C^p_B(C)$.

Proof. Let $0 \in A$ be a fixed element of A, and adjoin two elements θ, ω to A with actions $\omega s = \omega$ and $\theta s = 0$. Then $C^p_B(A) = A \cup \{\theta\}$ where $B = A \cup \{\theta, \omega\}$. Hence C^p is neither discrete nor trivial. Also, it is not minimal. Because, adjoining two elements θ, ω to an S-act C with actions $\omega s = \theta$ and $\theta s = \theta$, and taking $A = C \cup \{\theta\}$, $B = C \cup \{\theta, \omega\}$, we get $C \subset A \subset B$, and $C^p_B(A) = B$ while $C^p_B(C) = C$.

Another monomorphism which corresponds to this closure operator, and is of main interest in this paper, is defined as follows:

Definition 2.7. An S-act A is said to be C^p*-pure* in an extension B of A if $C^p(A) = C^d(A)$.

Remark 2.2. Let A_i be a family of subacts of A,

(1) If $\bigcap A_i$ is s-pure in A then $C^p_A(\bigcap A_i) = \bigcap C^p_A(A_i)$.

(2) If for every $i \in I$, A_i is s-pure in A and $C^p_A(\bigcap A_i) = \bigcap C^p_A(A_i)$ then $\bigcap A_i$ is s-pure in A.

Note 2.2. For $A \leq B$, we have $A \leq C^p(A) \leq C^d(A) \leq B$. So, if A is C^p-dense in B, then $C^p(A) = C^d(A) = B$ and so A is C^d-dense as well as C^p-pure. Similarly, if A is C^d-closed in B, then $A = C^p(A) = C^d(A)$ and hence A is C^p-closed as well as C^p-pure.

The following result, the proof of which is straightforward, is what we promised in the beginning of this section.

Theorem 2.3. *The following are equivalent:*

(1) *A is s-pure in B.*

(2) *A is C^p-pure in B.*

Lemma 2.4. (1) *Any retraction is s-pure.*

(2) *Any C^p-dense monomorphism is a retraction.*

Proof. : (1) Let $A \hookrightarrow B \xrightarrow{\pi} A = id_A$ be a retraction and $S \xrightarrow{k} A \hookrightarrow B = \lambda_b$ for some $b \in B$. It is clear that $k = \lambda_{\pi(b)}$.

(2) Let $A \hookrightarrow B$ be a C^p-dense subact. Then, by ?? it is C^p-pure as well as C^d-dense subact. So, for every $b \in B$ there exists $a_b \in A$ such that $\lambda_b = \lambda_{a_b}$. Now, for every $b \in B - A$ choose and fix such an $a_b \in A$. Define $\pi : B \to A$ by

$$\pi(x) = \begin{cases} x, & \text{if } x \in A \\ a_x, & \text{if } x \notin A \end{cases}$$

Then, clearly π is a retraction. It is a homomorphism because it is a homomorphism on A, and for $x \in B - A$, $s \in S$, we have $xs \in A$ and so $\pi(xs) = xs = a_x s = \pi(x)s$.

3. Categorical Properties of s-Pure Monomorphisms

In this section we investigate the categorical and algebraic properties, regarding composition, limits, and colimits, of the category **Act-S** with respect to the class $\mathcal{M}_p$ of sequentially pure monomorphisms. We have divided the section into three subsections as follows:

3.1. Composition properties of s-pure monomorphisms

In this subsection we investigate some properties of the class $\mathcal{M}_p$ which are mostly related to the composition of pure monomorphisms. These properties and the ones given in the next two subsections are what normally used to study injectivity with respect to a class of monomorphisms (see [1, 17])

Lemma 3.1. *The class $\mathcal{M}_p$ is:*

(1) *Isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.*

(2) *Composition closed; that is, if $f : A \to B$ and $g : B \to C$ belong to $\mathcal{M}_p$, then gf also belongs to $\mathcal{M}_p$.*

(3) *Left cancellable; that is, if $gf \in \mathcal{M}_p$ then $f \in \mathcal{M}_p$.*

Proof. We just prove (2), which may be less clear. For convenience and without loss of generality, we consider f and g to be s-pure inclusions. Let $S \xrightarrow{k} A \xhookrightarrow{f} B \xhookrightarrow{g} C = \lambda_c$, for some $c \in C$. Since g is s-pure, there is an element $b \in B$ such that $fk = \lambda_b$. Now, the s-purity of f provides an element $a \in A$ with $k = \lambda_a$.

Theorem 3.1. *The following are equivalent:*

(1) *S has a left identity.*

(2) *Every monomorphism is s-pure.*

(3) *S is s-pure in S^1.*

(4) *$\mathscr{M}_p$ is right cancellable; that is, for monomorphisms f and g, if gf is s-pure then g is s-pure.*

(5) *For morphisms f and g, if f is an s-pure monomorphism, g is an epimorphism, and gf is a monomorphism, then gf is s-pure. CATEGORICAL PROPERTIES OF*

Proof. (1)⇒(2,3,4,5): Clearly if S has a left identity, then every monomorphism is s-pure. So, (1) implies (2), (3), (4), and (5).

(2)⇒(3) is clear.

(3)⇒(1) Since $1 \in C^d_{S^1}(S)$, there exists $e \in S$ with $\lambda_e = \lambda_1$. This shows that e is a left identity of S.

(4)⇒(3) Use the fact that the empty set is s-pure in every right S-act, and apply (3) to $\emptyset \overset{f}{\hookrightarrow} S \overset{g}{\hookrightarrow} S^1$.

(5)⇒(3) Consider the natural homomorphisms $S \xrightarrow{\tau} S \sqcup S^1 \xrightarrow{\pi} S^1$. By Lemma 2.1, τ is s-pure. Since π is an epimorphism and $\pi\tau$ (the inclusion map) is one-one, by (5), $\pi\tau$ is s-pure.

As the above theorem shows, $\mathscr{M}_p$ is not generally right cancellable. But for some semigroups, regardless of having a left identity, some special monomorphisms may be cancelled from the right. See the following:

Lemma 3.2. *If $S^2 = S$, $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$ are monomorphisms, f is s-dense, and gf is s-pure, then g is s-pure.*

Proof. Without loss of generality, we again assume that f and g are inclusions. Let $c \in C$ be such that $cS \subseteq B$. So, since f is s-dense, we get $(cS)S \subseteq A$. Now, since $S^2 = S$ and gf is s-pure, we get an $a \in A \subseteq B$ with $\lambda_c = \lambda_a$, which proves that g is s-pure.

Definition 3.1. Let $\mathscr{E}$ be a class of homomorphisms. We say that **Act-S** has $(\mathscr{E}, \mathscr{M}_p)$ diagonalization property if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ f\downarrow & & \downarrow g \\ C & \xrightarrow{m} & D \end{array}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}_p$ there exists a unique diagonal S-map $d : B \to C$ such that $de = f$ and $md = g$.

Proposition 3.1. **Act-S** *has $(\mathscr{E}, \mathscr{M}_p)$ diagonalization property, for $\mathscr{E}$ the class of all epimorphisms.*

Proof. Consider the diagram given in the above definition. First, we see that $Ker\ e \subseteq Ker\ f$. Let $e(a) = e(a')$ and so $ge(a) = ge(a')$. Thus, $mf(a) = mf(a')$, and so $f(a) = f(a')$, since m is a monomorphism. Then, by The Decomposition Theorem (which holds since S-acts form an equational class), there exists a unique S-map $d : B \to C$ with $de = f$ (given by $d(b) = f(a)$, where $e(a) = b$). It is also easily seen that $md = g$.

Recall that a category is said to have *unique* $(\mathscr{E}, \mathscr{M})$ *factorization property* if every morphism f can be uniquely represented as $f = me$ with $e \in \mathscr{E}$ and $m \in \mathscr{M}$, where $\mathscr{E}$, $\mathscr{M}$ are some classes of morphisms.

Remark 3.1. **Act-S** does not generally have unique $(\mathscr{E}, \mathscr{M}_p)$ factorization property, where $\mathscr{E}$ is the class of all epimorphisms. To see this, let S be a semigroup that does not have a left identity. Then, by Theorem 3.1, S is not s-pure in S^1. On the contrary, let the inclusion morphism $\tau : S \to S^1$ have an $(\mathscr{E}, \mathscr{M}_p)$-factorization $S \xrightarrow{e} A \xrightarrow{m} S^1$. Since me is a monomorphism, e is a monomorphism and hence an isomorphism. Thus, $\tau = m$ is s-pure which is a contradiction.

3.2. Limits of s-pure monomorphisms

In this subsection some of the categorical properties of s-pure monomorphisms related to limits are studied. The proof of the following is straightforward.

Proposition 3.2. (1) *$\mathscr{M}_p$ is closed under products.*

(2) *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a family of s-pure monomorphisms. Then their product homomorphism $h : A \to \prod_{\alpha \in I} B_\alpha$ is also an s-pure monomorphism.*

Note that the above result (2) is also true whenever for some (not necessarily all) $\alpha \in I$, f_α is an s-pure monomorphism.

Lemma 3.3. *In* **Act-S**, *pullbacks transfer s-pure monomorphisms if and only if S has a left identity.*

Proof. Necessity: By Theorem 3.1, it is enough to show that S is s-pure in S^1. Let E be an injective S-act and 0 be a zero element of E (see [13], Lemma III.1.7). Adjoin an element θ to E and define $\theta s = 0$ for all $s \in S$. Then, E is clearly s-pure in $E^\theta = E \cup \{\theta\}$. Taking a homomorphism $f : S^1 \longrightarrow E^\theta$ given by $f(s) = \theta s$ $(s \in S^1)$ we have the pullback diagram:

$$\begin{array}{ccc} S & \xrightarrow{\tau} & S^1 \\ f\downarrow & & \downarrow f \\ E & \xrightarrow{\tau'} & E^\theta \end{array}$$

where τ, τ' are inclusion morphisms. Then, since τ' is s-pure, we get that S is s-pure in S^1, by the hypothesis.

Sufficiency: Let S have a left identity. In this case, by Theorem 3.1, every monomorphism is s-pure and pullbacks clearly preserve monomorphisms.

Proposition 3.3. *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of s-pure monomorphisms. Then the homomorphism $f : A \to \varprojlim B_\alpha$ (existing by the universal property of limits) is an s-pure monomorphism.*

Proof. It is clear that f is one-one, because so is every f_α. Also, if $k : S \to A$ is an S-map with $fk = \lambda_x$ for some $x \in \varprojlim B_\alpha$, then for every α $f_\alpha k = \pi_\alpha f k = \lambda_{\pi_\alpha(x)}$, where $\pi_\alpha : \varprojlim B_\alpha \to B_\alpha$ is a limit morphism. So, $k = \lambda_a$ for some $a \in A$, since f_α is s-pure.

Note that, the above result is also true whenever for some (not necessarily all) $\alpha \in I$, f_α is an s-pure monomorphism.

3.3. Colimits of s-pure monomorphisms

In this subsection we investigate the colimit properties of s-pure monomorphisms.

Proposition 3.4. *The class $\mathcal{M}_p$ is closed under coproducts.*

Proof. Let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of s-pure monomorphisms and $f : \coprod A_\alpha \to \coprod B_\alpha$ be the coproduct (mono)morphism induced by all f_α. Let $k : S \to \coprod A_\alpha$ be a homomorphism such that $fk = \lambda_b$ for some $b \in \coprod B_\alpha$. Since $b \in B_\alpha$ for some $\alpha \in I$, $k(S) \subseteq B_\alpha$, and so $k(S) \subseteq A_\alpha$. Since A_α is s-pure in B_α, $k = \lambda_{a_\alpha}$ for some $a_\alpha \in A_\alpha$, which proves the result.

In the following proposition, suppose that every A_α has a fixed element 0.

Proposition 3.5. (1) *The class $\mathcal{M}_p$ is closed under direct sums.*
(2) *If S is a finitely generated semigroup, then $\oplus_{\alpha \in I} A_\alpha$ is s-pure in $\prod_{\alpha \in I} A_\alpha$.*

Proof. (1) Let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of s-pure monomorphisms, and $f : \oplus A_\alpha \to \oplus B_\alpha$ be the monomorphism induced by the product of f_α, s. Let $k : S \to \oplus A_\alpha$ be a homomorphism such that $k = \lambda_b$ for some $b = (b_\alpha)_{\alpha \in I} \in \oplus B_\alpha$. Let J be a finite subset of I such that for all $\alpha \notin J$, $b_\alpha = 0$. So, for every $\beta \in J$, $f_\beta p_\beta k = \lambda_{b_\beta}$, where $p_\beta : \prod A_\beta \to A_\beta$ is the projection map. Since each f_β is s-pure, there exists $a_\beta \in A_\beta$ such that $p_\beta k = \lambda_{a_\beta}$. Thus, $k = \lambda_{(a_\alpha)_{\alpha \in I}}$, where for all $\alpha \notin J$, $a_\alpha = 0$.

(2) Let $k : S \to \oplus A_\alpha$ be a homomorphism with $k = \lambda_a$ for some $a = (a_\alpha)_{\alpha \in I} \in \prod A_\alpha$, and $S = \cup_{i=1}^n t_i S^1$. So, since $k(t_i) \in \oplus A_\alpha$, $at_i = (a_\alpha t_i)_{\alpha \in I} \in \oplus A_\alpha$. Thus, for every i there exists a finite subset J_i of I such that for every $\alpha \notin J_i$, $a_\alpha t_i = 0$. Now considering the finite subset $J = \bigcup_{i=1}^n J_i \subseteq I$ and

$$b_\alpha = \begin{cases} a_\alpha, & \text{if } \alpha \in J \\ 0, & \text{if } \alpha \notin J \end{cases},$$

it is clear that $k = \lambda_b$ for $b = (b_\alpha)_{\alpha \in I}$.

Theorem 3.2. *For the following pushout diagram in* **Act-S**, *we have:*
(1) *If f is a monomorphism then h is a monomorphism.*

(2) *If f is s-pure then h is s-pure.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g\downarrow & & \downarrow h' \\ C & \xrightarrow{h} & Q \end{array}$$

Proof. (1) Recall that $Q = (B \sqcup C)/\theta$ where $\theta = \rho(H)$ and H consists of all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C$, $u_C : C \to B \sqcup C$ are coproduct injections. And $h = \pi u_C : C \to (B \sqcup C)/\theta$, $h' = \pi u_B : B \to (B \sqcup C)/\theta$, where $\pi : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Let $h(c) = h(c')$, $c, c' \in C$, and so $u_C(c)\rho(H)u_C(c')$. Thus $c = c'$, and the result is proved, or there exist $a_1, a_2, ...a_n \in A$, $s_1, s_2, ..., s_n \in S^1$ such that $c = g(a_1 s_1)$, $g(a_n s_n) = c'$, and

$$f(a_1s_1) = f(a_2s_2) \cdots f(a_{n-1}s_{n-1}) = f(a_ns_n)$$
$$g(a_2s_2) = g(a_3s_3) \cdots$$

and then, the fact that f is a monomorphism gives

$$a_1s_1 = a_2s_2, a_3s_3 = a_4s_4, ..., a_{n-1}s_{n-1} = a_ns_n.$$

Thus, we get

$$g(a_1s_1) = g(a_2s_2) = g(a_3s_3) = \cdots = g(a_ns_n)$$

and hence $c = c'$.

(2) Let $k : S \to C$ be a homomorphism such that $hk = \lambda_x$ for some $x \in Q$. Two cases may occur: (i) there exists $c \in C$ such that $x = [u_C(c)]_{\rho(H)}$. Therefore, $hk = \lambda_x = \lambda_{h(c)}$ and hence $k = \lambda_c$, since h is one-one. (ii) there exists $b \in B$ such that $x = [u_B(b)]_{\rho(H)}$. For every $s \in S$, $[u_C(k(s))]_{\rho(H)} = [u_B(bs)]_{\rho(H)}$, and so there exist elements $a_1^s, \cdots, a_n^s$ in A such that $u_B(bs) = u_B f(a_1^s)$, $u_C g(a_1^s) = u_C g(a_2^s)$, $u_B f(a_2^s) = u_B f(a_3^s), \cdots, u_C g(a_n^s) = u_C(k(s))$. Then, since f is a monomorphism, $a_2^s = a_3^s, a_4^s = a_5^s, \cdots$, and hence $u_C g(a_1^s) = u_C g(a_2^s) = u_C g(a_3^s) = \cdots = u_C g(_n^s) = u_C(k(s))$. Now, for all $s \in S$, $bs = f(a_1^s) \in f(A)$, and thus there exists $k_1 : S \to A$ with $k_1(s) = a_1^s$, and so $f k_1 = \lambda_b$. Then, since f is s-pure, there is an element $a \in A$ such that $k_1 = \lambda_a$. Therefore, $f(a_1^s) = f(k_1(s)) = f(as)$ and so $hg(a_1^s) = h' f(a_1^s) = h' f(as) = hg(as)$ which, since h is a monomorphism, yields $k(s) = g(a_1^s) = g(as) = g(a)s$, that is $k = \lambda_{g(a)}$, and h is s-pure.

Theorem 3.3. *Let I be a directed set which has a maximal element γ and $\{h_\alpha : A_\alpha \to B_\alpha |\ \alpha \in I\}$ be a directed family of s-pure monomorphisms. Then, the directed colimit homomorphism induced by $h : \varinjlim A_\alpha \to \varinjlim B_\alpha$ is s-pure .*

Proof. Let $(\varinjlim A_\alpha, f_\alpha), (\varinjlim B_\alpha, g_\alpha)$ be direct limits of the directed systems $((A_\alpha), (\psi_{\alpha\beta}))_{\alpha\le\beta\in I}$ and $((B_\alpha), (\varphi_{\alpha\beta}))_{\alpha\le\beta\in I}$ and suppose $\{h_\alpha : A_\alpha \to B_\alpha |\ \alpha \in I\}$ is a directed family of s-pure monomorphisms such that for every $\alpha \le \beta$, $f_\beta \psi_{\alpha\beta} = f_\alpha$ and $g_\beta \varphi_{\alpha\beta} = g_\alpha$. Then, for every $\alpha \le \beta$, $g_\beta h_\beta \psi_{\alpha\beta} = g_\beta \varphi_{\alpha\beta} h_\alpha = g_\alpha h_\alpha$, so $h = \varinjlim h_\alpha$ exists by the universal property of colimits. Consider $\varinjlim A_\alpha = A_\alpha/\rho$ and $\varinjlim B_\alpha = B_\alpha/\rho'$ as defined in section 1. Let

$h[a_\alpha]_\rho = h[a_\beta]_\rho$. Then, $[h_\alpha(a_\alpha)]_{\rho'} = g_\alpha h_\alpha(a_\alpha) = g_\beta h_\beta(a_\beta) = [h_\beta(a_\beta)]_{\rho'}$, and so there exists $\gamma \in I$ with $\gamma \geq \alpha, \beta$ and $\varphi_{\alpha\gamma} h_\alpha(a_\alpha) = \varphi_{\beta\gamma} h_\beta(a_\beta)$ which implies that $[a_\alpha]_\rho = [a_\beta]_\rho$, and so h is a monomorphism.

Now, let $k : S \to \varinjlim A_\alpha$ be a homomorphism such that $hk = \lambda_{[b_\alpha]_{\rho'}}$ and for $s \in S$, $k(s) = [a_{\alpha_s}]_\rho$, $a_{\alpha_s} \in A$. Notice that $\gamma \geq \alpha, \alpha_s$, for $s \in S$, and define $k_1 : S \to A_\gamma$ by $k_1(s) = \psi_{\alpha_s \gamma}(a_{\alpha_s})$. Then, since $f_\gamma k_1(s) = f_\gamma \psi_{\alpha_s\gamma}(a_{\alpha_s}) = f_{\alpha_s}(a_{\alpha_s}) = [a_{\alpha_s}]_\rho = k(s)$ and $f_\gamma k_1(st) = k(st) = k(s)t = f_\gamma k_1(s)t$, it follows that k_1 is a homomorphism, since f_γ is a monomorphism. Now, for every $s \in S$,

$$g_\gamma h_\gamma k_1(s) = hf_\gamma k_1(s) = hk(s) = [b_\alpha]s = g_\alpha(b_\alpha)s = g_\gamma \varphi_{\alpha\gamma}(b_\alpha s)$$

and then, since g_γ is a monomorphism, we have $h_\gamma k_1 = \lambda_{\varphi_{\alpha\gamma}(b_\alpha)}$. But, h_γ is s-pure and so $k_1 = \lambda_a$ for some $a \in A_\gamma$. Hence $k = \lambda_{f_\gamma(a)}$, because $f_\gamma k_1 = k$.

Corollary 3.1. *Let I be a directed set which has a maximal element γ and $\{h_\alpha : A \to B_\alpha | \alpha \in I\}$ be a directed family of s-pure monomorphisms. Then, the directed limit (colimit) of h_α,s is s-pure.*

Proof. Let $h : A \to \varinjlim_\alpha B_\alpha$ be a direct limit in **Act-S** of s-pure monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$, and consider $g_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha$ as in the definition. Recall that $h = \varinjlim_\alpha h_\alpha = g_\gamma h_\gamma = g_\alpha h_\alpha = g_\beta h_\beta = \dots$. It is clear that $h : A \to \varinjlim_\alpha B_\alpha$ is a directed colimit of the directed family $\{h_\alpha : id : A_\alpha \to A_\gamma \mid \alpha \in I - \{\gamma\}, A_\alpha = A\}$. Then, apply the above theorem to complete the proof.

Another condition which gives the above result is finitely generatedness of semigroup:

Theorem 3.4. *Let I be a directed set and S be a finitely generated semigroup. Then, the category* **Act**-*S has $\mathcal{M}_p$-directed colimits.*

Proof. Let $h : A \to \varinjlim_\alpha B_\alpha$ be a direct limit in **Act-S** of s-pure monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$, and consider $g_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha$ as in the definition. Recall that $h = \varinjlim_\alpha h_\alpha = g_\gamma h_\gamma = g_\alpha h_\alpha = g_\beta h_\beta = \dots$. Let $S = \cup_{i=1}^n t_i S^1$ and $k : S \to A$ be an S-map such that $hk = \lambda_{[b_\alpha]}$. Then, for every $1 \leq i \leq n$, $[h_\alpha k(t_i)] = [b_\alpha t_i]$, and so there exist $\gamma_i \in I$ such that $\psi_{\alpha\gamma_i}(h_\alpha k(t_i)) = \psi_{\alpha\gamma_i}(b_\alpha t_i)$. Now, let $\gamma \geq max\{\gamma_1, \cdots, \gamma_n\}$ and so for every $s \in S$, $h_\gamma(k(s)) = \psi_{\alpha\gamma}(b_\alpha)s$. Then, since h_γ is s-pure, $k = \lambda_a$ for some $a \in A$.

We say that *multiple pushouts* transfer s-pure monomorphisms if in multiple pushout $(P, A_\alpha \xrightarrow{h_\alpha} P)$ of a family of s-pure monomorphisms $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$, every h_α, $\alpha \in I$, is an s-pure monomorphism.

Theorem 3.5. *Multiple pushouts transfer s-pure monomorphisms.*

Proof. Let $(P, A_\alpha \xrightarrow{h_\alpha} P)$ be the multiple pushout of the family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of s-pure monomorphisms. We know that $P = \coprod A_\alpha / \rho(H)$ where $H = \{(f_\alpha(a), f_\beta(a)) \mid a \in A, \alpha, \beta \in I\}$ (we have taken the image of each element A_α under coproduct morphisms equal to itself). Let $h_\alpha(a) = h_\alpha(a')$, $a, a' \in A_\alpha$, and so there exist $p_1, p_2, ..., p_n, q_1, q_2, ..., q_n \in A, s_1, s_2, ..., s_n \in$

S^1 where for $i = 1, ..., n, (p_i, q_i) \in H \cup H^{-1}$ and such that $a = p_1 s_1, q_1 s_1 = p_2 s_2, q_2 s_2 = p_3 s_3, ..., q_n s_n = a'$. Then, $a = f_\alpha(a_1)s_1$ and there exists $\beta \in I$ such that $f_\beta(a_1)s_1 = f_\beta(a_2)s_2$. Then, since f_β is a monomorphism, $a_1 s_1 = a_2 s_2$. Continuing this process we get that $a_1 s_1 = a_2 s_2 = ... = a_n s_n$, and therefore $a = a'$. Now, let $k : S \to A_\alpha$ be a homomorphism such that $h_\alpha k = \lambda_{[p]}$. If $p \in A_\alpha$ then, since h_α is a monomorphism, $k = \lambda_p$. If $p \in A_\beta,\ \beta \neq \alpha$, then for every $s \in S$, $ps = f_\beta(a_1)s_1$ and thus for every $s \in S,\ ps \in f_\beta(A)$. So, there exists $a \in A$ such that $k = \lambda_{f_\beta(a)}$.

Corollary 3.2. *Every multiple pushout of s-pure monomorphisms* (*the diagonal maps on the multiple pushout diagram*) *is an s-pure monomorphism.*

Proof. Apply Lemma 3.1(2) and the above theorem.

Definition 3.2. The category **Act-S** has:

(1) $\mathscr{M}_p$*-bounds* if for any small (and non-empty) family $(h_\alpha : A \to B_\alpha)_{\alpha \in I}$ of $\mathscr{M}_p$-morphisms there is an $\mathscr{M}_p$-morphism $h : A \to B$ which factorizes through all h_α,s.

(2) $\mathscr{M}_p$*-amalgamation property* if in (i) h factorizes through all h_α,s by $\mathscr{M}_p$ maps.

The above corollary gives that:

Proposition 3.6. **Act-S** *has* $\mathscr{M}_p$*-amalgamation property and so also has* $\mathscr{M}_p$*-bound.*

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