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A Note on Prüfer Modules

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Abstract. In this paper we characterize Prüfer modules and Dedekind modules.

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1. Introduction

Throughout this paper R denotes a commutative ring with identity and M denotes a unital R-module. $L(R)$ ($L(M)$) denotes the lattice of all ideals of R (submodules of M). For any two submodules N and K of M, the ideal $\{a \in R \mid aK \subseteq N\}$ will be denoted by $(N : K)$. Thus $(O : M)$ is the annihilator of M. M is said to be a *faithful module* if $(O : M)$ is the zero ideal of R. M is said to be a *multiplication module* [4] if every submodule of M is of the form IM, for some ideal I of R. According to [7], a submodule N of M is called meet-quasi-cyclic (or meet principal in the sense of [1, 3]) if $(B \cap (K : N))N = BN \cap K$ for all ideals B of R and for all submodules K of M; N is called weak-join-quasi-cyclic if $(BN) : N = (0 : N) + B$ for all ideals B of R; N is called join-quasi-cyclic (or join principal in the sense of [1] and [3]) if $(K + BN) : N = (K : N) + B$ for all ideals B of R and for all submodules K of M. N is called quasi-cyclic [7] (or principal in the sense of [1] and [3])if N is both meet-quasi-cyclic and join-quasi-cyclic. Note that quasi-cyclic submodules have been studied in [1], [3] and [7].

For any $a \in R$, the principal ideal generated by a is denoted by (a). Recall that an ideal I of R is called a *multiplication ideal* if for every ideal $J \subseteq I$, there exists an ideal K with $J = KI$. An ideal I of R is called weak join principal if $(AI : I) = A + (0 : I)$ for all $A \in L(R)$. I is called join principal if $(A + BI) : I = (A : I) + B$, for all $A, B \in L(R)$. An ideal I of R is called a *quasi-principal ideal* [8, Exercise 10, Page 147] (or a principal element of $L(R)$ [9]) if it satisfies the identities

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(i) $(A \cap (B : I))I = AI \cap B$ and

(ii) $(A + BI) : I = (A : I) + B$, for all A, $B \in L(R)$. Obviously, every quasi-principal ideal is a multiplication ideal. Quasi-principal ideals have been studied in [2, 5, 9].

Let S be the set of all non-zero divisors of R and let $T = \{t \in S : tm = 0$ for some $m \in M$ implies $m = 0\}$. Let R_T be the localization of R at T. For any non-zero submodule N of M, let $N^{-1} = \{x \in R_T : xN \subseteq M\}$. It is easily seen that N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $N^{-1}N \subseteq M$. Following [10], N is an invertible submodule of M if $N^{-1}N = M$. Following [10], an R module M is called a Dedekind module (Prüfer module) if every non-zero (finitely generated) submodule of M is invertible. Dedekind modules and Prüfer modules have been extensively studied in [1] and [10].

In [1, Theorem 2.3], it is proved that if R is an integral domain and M is a faithful multiplication R-module, then M is a Prüfer module if and only if every finitely generated submodule of M is principal. In this paper we prove that if M is a non zero faithful multiplication R-module, then R is a Prüfer module if and only if R is an integral domain and every finitely generated submodule of M is join-quasi-cyclic (i.e., join principal). Next we show that if M is a non zero faithful multiplication R-module, then R is a Dedekind module if and only if R is an integral domain and every submodule of M is a finitely generated join-quasi-cyclic submodule of M.

For general background and terminology, the reader is referred to [8].

2. Prüfer Modules and Dedekind Modules.

In this paper we establish some new characterizations for Prüfer modules and Dedekind modules. We shall begin with the following lemmas.

Lemma 1. *Suppose M is a non zero faithful finitely generated weak-join-quasi-cyclic R-module and B is an ideal of R. If BM is weak-join-quasi-cyclic (join-quasi-cyclic), then B is weak join principal (join principal).*

Proof. Let $A \in L(R)$. Since M is faithful and weak-join-quasi-cyclic, we have $(AB : B) = (ABM : BM)$. As BM is weak-join-quasi-cyclic, we have $(ABM : BM) = A + (0 : BM) = A + ((0 : M) : B) = A + (0 : B)$. Therefore B is weak join principal.

Let $A, C \in L(R)$. Since M is faithful and weak-join-quasi-cyclic, it follows that $((AB + C) : B) = ((ABM + CM) : BM)$. As BM is join-quasi-cyclic, we have $((ABM + CM) : BM) = A + (CM : BM) = A + (C : B)$ since M is faithful and weak-join-quasi-cyclic. Therefore B is join principal.

Lemma 2. *Suppose M is a non zero faithful finitely generated weak-join-quasi-cyclic R-module. Suppose R is an integral domain and B is a finitely generated ideal of R. If BM is weak-join-quasi-cyclic, then B is quasi-principal.*

Proof. By lemma 1, B is weak join principal, so by [2, Theorem 4], B is quasi-principal.

Lemma 3. *Suppose R is an integral domain and M is a non zero faithful finitely generated R-module. If every finitely generated submodule of M is weak-join-quasi-cyclic, then R is a Prüfer domain.*

Proof. Let I be a finitely generated ideal of R. Then IM is finitely generated, so IM is weak-join-quasi-cyclic. By Lemma 2, I is quasi-principal and hence R is a Prüfer domain [8, Page 147, Ex. 10(e)].

Lemma 4. *Suppose R is an arithmetical ring and M is a non zero finitely generated R-module. Then every finitely generated submodule of M is join-quasi-cyclic.*

Proof. Let N be a finitely generated submodule of M. It is enough to show that N is locally join-quasi-cyclic. Assume that R is a valuation ring(i.e., any two ideals are comparable). Let $A \in L(R)$ and $B \in L(M)$. Clearly, $A + (B : N) \subseteq ((AN + B) : N)$. Let $a \in ((AN + B) : N)$. Then $aN \subseteq AN + B$. We have either $(a) \subseteq A$ or $A \subseteq (a)$. If $(a) \subseteq A$, then we are through. Suppose $A \subset (a)$. As (a) is a multiplication ideal, it follows that $A = J(a)$ for some proper ideal J of R. So $aN \subseteq J(a)N + B$, so by Nakayama's lemma $aN \subseteq B$ and hence $a \in (B : N)$. Therefore N is join-quasi-cyclic.

Lemma 5. *Suppose R is an integral domain and M is a non zero faithful finitely generated R-module. Then R is a Prüfer domain if and only if every finitely generated submodule of M is join-quasi-cyclic.*

Proof. The proof of the lemma follows from Lemma 3 and Lemma 4.

Theorem 1. *Suppose M is a non zero faithful multiplication R-module. Then M is a Prüfer module if and only if R is an integral domain and every finitely generated submodule of M is join-quasi-cyclic.*

Proof. Suppose M is a Prüfer module. Then by [10, Theorem 3.6], R is a Prüfer domain. As R is an integral domain and M is a non zero faithful multiplication R-module, by [6, Proposition 3.4], M is finitely generated. Again by Lemma 5, every finitely generated submodule of M is join-quasi-cyclic. The converse part follows from [6, Proposition 3.4], Lemma 5 and [10, Theorem 3.6].

Theorem 2. *Suppose M is a non zero faithful multiplication R-module. Then M is a Dedekind module if and only if R is an integral domain and every submodule of M is a finitely generated join-quasi-cyclic submodule of M.*

Proof. The proof of the theorem follows from Theorem 1 and [1, Theorem 3.4].

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