Majorization for Certain Analytic Functions

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Abstract. In this paper two subclasses \( S_\delta^p(\gamma,A,B) \) and \( C_\delta^p(\gamma,A,B) \) of p-valently starlike and p-valently convex functions of complex order \( \gamma \neq 0 \) in the open unit disk \( U \) are introduced and for these classes several majorization problems are discussed.

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1. Introduction and Definitions

Definition 1 ([see 5]). Let the functions \( f(z) \) and \( g(z) \) be analytic in the open unit disk

\[ U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}. \]

We say that \( f(z) \) is majorized by \( g(z) \) and write

\[ f(z) \ll g(z) \]  \hspace{1cm} (1)

if there exists a function \( \phi(z) \) analytic in \( U \), such that

\[ |\phi(z)| \leq 1 \text{ and } f(z) = \phi(z)g(z). \]  \hspace{1cm} (2)

Also, we say that \( f(z) \) is subordinate to \( g(z) \) and write

\[ f(z) \prec g(z) \]

if there exist a function \( w(z) \) analytic in \( U \), such that

\[ w(0) = 0, |w(z)| \leq |z| \text{ and } f(z) = g(w(z)). \]
Definition 2 ([see 8]). The fractional derivative of order $\delta$ is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\delta}} d\zeta \quad (0 \leq \delta < 1)$$

(3)

where $f(z)$ is an analytic function in a simply connected region of the $z$–plane containing the origin and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 3 ([see 8]). Under the hypotheses of definition 2, the fractional derivative of order $(n+\delta)$ is defined by

$$D_z^{n+\delta} f(z) = \frac{d}{dz^n} D_z^\delta f(z).$$

(4)

Several majorization problems investigated by Altıntaş and Owa [1], Altıntaş et al. [2] and [3].

Let $A_p$ denote the class of functions $f$ normalized by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (P \in \mathbb{N} = \{1, 2, 3, \ldots\})$$

which are analytic and $p$–valent in $U$. Also let a function $f \in A_p$ is said to be in the class $S_{p,q}^{\delta}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma} \left( \frac{z f^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)} - p + q + \delta \right) < \frac{1+AZ}{1+BZ}$$

(5)

where $\gamma \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 \leq \delta < 1$, $-1 \leq B < A \leq 1$ and

$$|\gamma(A-B) + (p-q-\delta)B| \leq |p-q-\delta|.$$

Furthermore a function $f \in A_p$ is said to be in the class $C_{p,q}^{\delta}(\gamma, A, B)$ if and only if

$$1 + \frac{1}{\gamma} \left( 1 + \frac{z f^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta \right) \leq \frac{1+AZ}{1+BZ}$$

(6)

where $\gamma \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, $0 \leq \delta < 1$, $-1 \leq B < A \leq 1$ and

$$|\gamma(A-B) + (p-q-\delta)B| \leq |p-q-\delta|.$$

We have the following relationships (from [3, 11, 2], respectively)

$$S_{p,q}^0(\gamma, 1, -1) = S_{p,q}(\gamma).$$

$$C_{p,q}^0(\gamma, 1, -1) = C_{p,q}(\gamma).$$
were introduced by Robertson in [9].

\[ S^0_{p,0}(\gamma, 1, -1) = S(\gamma) \]  

and 

\[ C^0_{p,0}(\gamma, 1, -1) = C(\gamma). \]

\( S(\gamma) \) and \( C(\gamma) \) were considered by Nasr and Aouf in [6].

\[ S^0_{p,0}(1 - \alpha, 1, -1) = S^*(\alpha) \]  

and 

\[ C^0_{p,0}(1 - \alpha, 1, -1) = C(\alpha) \]

denote respectively the class of starlike and convex functions of order \( \alpha, (0 \leq \alpha < 1) \) which were introduced by Robertson in [9].

### 2. Majorization Problems for the Class \( S^\delta_{p,q}(\gamma, A, B) \)

We begin by proving.

**Theorem 1.** Let the function \( f(z) \) be in the class \( A_p \) and suppose that \( g \in S^\delta_{p,q}(\gamma, A, B) \). If \( f^{(q+\delta)}(z) \) is majorized by \( g^{(q+\delta)}(z) \) in \( U \) for \( q \in \mathbb{N}_0 \) and \( 0 \leq \delta < 1 \), then

\[
\left| f^{(q+\delta+1)}(z) \right| \leq \left| g^{(q+\delta+1)}(z) \right| \quad (|z| \leq r_1)
\]

where \( r_1 = r_1(p, q, \delta, \gamma, A, B) \) is the smallest positive root of the equation

\[
\gamma(A - B) + (p - q - \delta)B \left| r^3 - (p - q - \delta + 2|B|)r^2 - (\gamma(A - B) + (p - q - \delta)B) \right| + 2|B| + p - q - \delta = 0
\]

where \( p \in \mathbb{N}, q \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \delta < 1 \) and

\[
\left| \gamma(A - B) + (p - q - \delta)B \right| \leq |p - q - \delta|.
\]

**Proof.** Since \( g \in S^\delta_{p,q}(\gamma, A, B) \), we obtain from (5)

\[
1 + \frac{1}{\gamma} \left( \frac{z g^{(q+\delta+1)}(z)}{g^{(q+\delta)}(z)} - p + q + \delta \right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}
\]

where

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| \leq |z| \quad (z \in U).
\]

From (9) we readily obtain

\[
\frac{z g^{(q+\delta+1)}(z)}{g^{(q+\delta)}(z)} = \frac{p - q - \delta + (\gamma(A - B) + (p - q - \delta)B) \omega(z)}{1 + B\omega(z)}.
\]

Using (10) in (11) we find

\[
\left| g^{(q+\delta)}(z) \right| \leq \frac{(1 + |B||z|)|z|}{p - q - \delta - (\gamma(A - B) + (p - q - \delta)B)|z|} \left| \frac{z g^{(q+\delta+1)}(z)}{g^{(q+\delta)}(z)} \right|.
\]
Hence, by setting $\rho$ in Corollary 1 leads us to the inequality then the function $f^{(q+\delta)}(z)$ is majorized by $g^{(q+\delta)}(z)$ from (2) we have
\[
f^{(q+\delta+1)}(z) = \phi(z)g^{(q+\delta+1)}(z) + \phi'(z)g^{(q+\delta)}(z),
\]
(13)
$\phi(z)$ is satisfies the inequality [cf. Nehari 7, p. 168]:
\[
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} (z \in U)
\]
(14)
and using (12) and (14) in (13), we get
\[
|f^{(q+\delta+1)}(z)| \leq |\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|^2} (1 + |B||z|)|z| p - q - \delta - |\gamma(A - B) + (p - q - \delta)B| |z| g^{(q+\delta+1)}(z)
\]
(15)
which, upon setting
\[
|z| = r, |\phi(z)| = \rho \quad (0 \leq \rho \leq 1)
\]
leads us to the inequality
\[
|f^{(q+\delta+1)}(z)| \leq \frac{\theta(\rho)}{(1 - r^2) [p - q - \delta - |\gamma(A - B) + (p - q - \delta)B| r]} g^{(q+\delta+1)}(z)
\]
(16)
where
\[
\theta(\rho) = -(r + |B|r^2)\rho^2 + (1 - r^2)p - q - \delta - |\gamma(A - B) + (p - q - \delta)B| r ) \rho + (r + |B|r^2)
\]
(17)
takes on its maximum value at $\rho = 1$ with $r = r_1(p, q, \delta, \gamma, A, B)$ gives by (8) if
\[
0 \leq \sigma \leq r_1(p, q, \delta, \gamma, A, B)
\]
then the function $\wedge(\rho)$ defined by
\[
\wedge(\rho) = -(\sigma + \sigma^2 |B|)\rho^2 + (1 - \sigma^2) [p - q - \delta - |\gamma(A - B) + (p - q - \delta)B| \sigma ] \rho + (\sigma + \sigma^2 |B|)
\]
(18)
is an increasing function on the interval $0 \leq \rho \leq 1$ so that
\[
\wedge(\rho) \leq \wedge(1) = (1 - \sigma^2) [p - q - \delta - |\gamma(A - B) + (p - q - \delta)B| \sigma ]
\]
(0 \leq \rho \leq 1; 0 \leq \sigma \leq r_1(p, q, \delta, \gamma, A, B)).

Hence, by setting $\rho = 1$ in (16), we conclude that Theorem 1 holds true for $|z| \leq r_1(p, q, \delta, \gamma, A, B)$ is given by (8). This completes the proof of Theorem 1.

**Corollary 1** ([see 3]). Let the function $f(z)$ be in the class $A_p$ and suppose that $g \in S_{p,q}^0(\gamma, 1, -1)$. If $f^{(q)}(z)$ is majorized by $g^{(q)}(z)$ in $U$, then
\[
|f^{(q+1)}(z)| \leq |g^{(q+1)}(z)| \quad (|z| \leq R_1)
\]
where
\[
R_1 = R_1(p, q, \delta) = \frac{k - \sqrt{k^2 - 4(p - q)2\gamma - p + q}}{2|2\gamma - p + q|}
\]
\[(k = p - q + 2 + |2\gamma - p + q|, p \in \mathbb{N}, q \in \mathbb{N} \setminus \{0\}).
\]

**Proof.** If we set \(\delta = 0, A = 1, B = -1\) in Theorem 1, then
\[
|f^{(q+1)}(z)| \leq |g^{(q+1)}(z)| \quad |z| \leq R_1
\]
where \(R_1 = R_1(p, q, \delta)\) is the smallest positive root of the equation
\[
|2\gamma - p + q| r^3 - (p - q + 2)r^2 - [|2\gamma - p + q| + 2]r + p - q = 0
\]
r = -1 is the root of the above equation and we obtain
\[
|2\gamma - p + q| r^2 - (|2\gamma - p + q| + p - q + 2)r + p - q = 0.
\]
and the positive root of the equation (20) is \(R_1 = R_1(p, q, \delta)\).

**Corollary 2** ([see 2]). Let the function \(f(z)\) be in the class \(A_1\) and suppose that \(g \in S_{1,0}^0(\gamma, 1, -1)\). If \(f(z)\) is majorized by \(g(z)\) in \(U\), then
\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq R_2)
\]
where
\[
R_2 = R_2(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}
\]

**Corollary 3** ([see 5]). Let \(f(z)\) be in the class \(A_1\) and suppose that \(g(z) \in S_{1,0}^0(1, 1, -1)\). If \(f(z)\) is majorized by \(g(z)\) in \(U\), then
\[
|f'(z)| \leq |g'(z)| \quad (|z| \leq R_3)
\]
where \(R_3 = 2 - \sqrt{3}\).

3. Majorization Problems for the Class \(C_{p,q}^\delta(\gamma, A, B)\).

The proof Theorem 2 is based upon the following Lemmas.

**Lemma 1** ([see 10, Theorem 1]). If \(f \in C_{p,q}^\delta(\gamma, A, B) (\gamma \in \mathbb{C} \setminus \{0\})\) then
\[
Re \left[ 1 + \frac{1}{\gamma} \left( \frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1 \right) \right] > \frac{1-A}{1-B}
\]
Proof. If \( f \in C_{p,q}^\delta (\gamma, A, B) \) then we have from (6)
\[
1 + \frac{1}{\gamma} \left( \frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1 \right) = \frac{1 + Aw(z)}{1 + Bw(z)}
\]
(22)
where \( w(0) = 0 \) and \( |w(z)| \leq |z|, (-1 \leq B < A \leq 1) \). We let
\[
h(z) = \frac{1 + Aw(z)}{1 + Bw(z)}
\]
and
\[
h(z) = u + iv, \quad |w(z)|^2 = \frac{|h(z) - 1|^2}{|A - Bh(z)|} \leq 1
\]
and
\[
(1 - B^2)u^2 - 2(1 - AB)u + 1 - A^2 \leq 0
\]
(24)
from (24) implies that
\[
\frac{1 - A}{1 - B} \leq Reh(z) = u \leq \frac{1 + A}{1 + B}
\]
(25)
The following lemma is proved in [3] for \( \delta = 0 \).

**Lemma 2.** If \( f \in C_{p,q}^\delta (\gamma, A, B) (\gamma \in C \setminus \{0\}) \) then \( f \in S_{p,q}^\delta (\frac{\gamma}{2}, A, B) \) that is
\[
C_{p,q}^\delta (\gamma, A, B) \subset S_{p,q}^\delta (\frac{\gamma}{2}, A, B)
\]
(26)

**Proof.** We know that all convex function in \( U \) is starlike of order \( \frac{1}{2} \) in \( U \), [see 4, p. 7] or, equivalently
\[
Re[1 + \frac{zf''(z)}{f'(z)}] > 0 \Rightarrow Re[\frac{zf'(z)}{f(z)}] > \frac{1}{2}
\]
(27)
If we let
\[
Re[1 + \frac{zf''(z)}{f'(z)}] > \alpha \text{ for } f(z) \longrightarrow f^{(q+\delta)}(z),
\]
and using Lemma 1, we have
\[
Re[1 + \frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1] > \frac{1 - A}{1 - B}
\]
(28)
or
\[
Re[1 + \frac{1}{1 - \alpha} \left( \frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1 \right) > 0.
\]
(29)
This implies that
\[
1 + \frac{1}{1 - \alpha} \left( \frac{zf^{(q+\delta+2)}(z)}{f^{(q+\delta+1)}(z)} - p + q + \delta + 1 = \frac{1 - w(z)}{1 + w(z)}
\]
(30)
So, we have

\[ 1 + \frac{1}{\gamma} f^{(q+\delta+2)}(z) - (p + q + \delta + 1) = \frac{\gamma + (\gamma - 2 + 2\alpha) w(z)}{\gamma(1 + w(z))}. \] (31)

On the other hand we know that

\[ \frac{zf'(z)}{f(z)} > \alpha \Rightarrow \text{Re}(1 + \frac{1}{1 - \alpha} \frac{zf'(z)}{f(z)}) > 0. \] (32)

Similarly using (27) and (29) we obtain the following relations.

\[ [1 + \frac{1}{1 - \alpha} \left( \frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)} - p + q + \delta \right)] > \frac{1}{2}, \] (33)

\[ 1 + \frac{1}{1 - \alpha} \left( \frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)} - p + q + \delta \right) = \frac{1}{1 + w(z)}, \] (34)

\[ 1 + \frac{2}{\gamma} \frac{zf^{(q+\delta+1)}(z)}{f^{(q+\delta)}(z)} - p + q + \delta = \frac{\gamma + (\gamma - 2 + 2\alpha)}{\gamma(1 + w(z))}. \] (35)

The inclusion property (26) is easily seen that from (31) and (35).

Upon replacing \( \gamma \) in Theorem 1 by \( \frac{1}{2} \gamma \), if we apply Lemma 2 we have,

**Theorem 2.** Let the function \( f(z) \) be in the class \( A_p \) and suppose that \( g(z) \in C^\delta_{p,q}(\gamma,A,B) \). If \( f^{(q+\delta)}(z) \) is majorized by \( g^{(q+\delta)}(z) \in U \), for \( p \in \mathbb{N} q \in \mathbb{N}_0 \) and \( 0 \leq \delta < 1 \) then

\[ \left| f^{(q+\delta+1)}(z) \right| \leq \left| g^{(q+\delta+1)}(z) \right| \quad (|z| \leq r_2) \] (37)

where \( r_2 = r_2(p,q,\delta,\gamma,A,B) \) is the smallest positive root of the equation

\[ \left| \frac{1}{2} \gamma(A - B) + (p - q - \delta)B \right| r^3 - (p - q - \delta + 2 |B| r^2 - \left[ \frac{1}{2} \gamma(A - B) + (p - q - \delta) |B| \right. \] \[ + 2] r + p - q - \delta = 0 \]

where \( p \in \mathbb{N}, q \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \delta < 1, \) and

\[ \left| \frac{1}{2} \gamma(A - B) + (p - q - \delta)B \right| \leq |p - q - \delta|. \]

**Corollary 4 ([see 3]).** Let the function \( f(z) \) be in the class \( A_p \) and suppose that \( g(z) \in C^0_{p,q}(\gamma,1,-1) \). If \( f^{(q)}(z) \) is majorized by \( g^{(q)}(z) \) in \( U \), then

\[ \left| f^{(q+1)}(z) \right| \leq \left| g^{(q+1)}(z) \right| \quad (|z| \leq R_1) \]
where
\[ R_1 = R_1(p, q, \delta) = \frac{\mu - \sqrt{\mu^2 - 4(p-q)\gamma - p+q}}{2|\gamma - p+q|} \]
\[ \mu = 2 + p - q + |\gamma - p + q|, p \in \mathbb{N}, q \in \mathbb{N}_0, \gamma \in \mathbb{C} \setminus \{0\} \]

**Proof.** We let \( \delta = 0, A = 1, B = -1 \) in Theorem 2.

**Corollary 5** ([see 2]). Let the function \( f(z) \) be in the class \( A_1 \) and suppose that \( g(z) \in C^0_{1,0}(1, 1, -1) \). If \( f(z) \) is majorized by \( g(z) \) in \( U \), then
\[ |f'(z)| \leq |g'(z)| \quad (|z| \leq R_2) \]

where
\[ R_2 = R_2(\gamma) = \frac{3 + |\gamma - 1| - \sqrt{9 - 2|\gamma - 1| + |\gamma - 1|^2}}{2|\gamma - 1|} \]

**Proof.** We let \( p = 1, q = 0, \delta = 0, A = 1, B = -1 \) in Theorem 2.

**Corollary 6** ([see 5]). Let the function \( f(z) \) be in the class \( A_1 \) and suppose that \( g(z) \in C^0_{1,0}(1, 1, -1) \). If \( f(z) \) is majorized by \( g(z) \) in \( U \), then
\[ |f'(z)| \leq |g'(z)| \quad (|z| \leq \frac{1}{3}) \]

**Proof.** We let limit for \( \gamma \rightarrow 1 \) in Corollary 5 or \( \gamma \rightarrow \frac{1}{2}\gamma \) in Corollary 1.

**References**


