Rate of Convergence in Sobolev Space

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Abstract. In this paper, a new theorem on degree of approximation in $L^2_p(\Omega)$ Sobolev space of integrable functions of two variables by Bernstein-Chlodowsky polynomials on an unbounded triangular domain is studied. Also by using the $K$- functional of Peetre the order of approximation are established.

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1. Introduction

The aim of this paper is to study the problem on degree of the approximation of function of two variables of $f \in L^2_p(\Omega)$ by means of Bernstein-Chlodowsky polynomials in a triangular domain extending infinity, where $\Omega = \lim_{n \to \infty} \Delta b_n$, $\Delta b_n = \{(x,y) : x \leq 0, y \geq 0, x + y \leq b_n\}$ and $(b_n)$ is a sequence of increasing positive number, such that:

$$\lim_{n \to \infty} b_n = \infty, \lim_{n \to \infty} \frac{b_n}{n} = 0. \quad (1)$$

Some properties of approximation of functions of two variable by Bernstein-Chlodowsky polynomials was proven in [1]-[5] and [7]. In addition, convergence of Bernstein-Chlodowsky polynomials of two variables were investigated on a triangular domain in [6] and [7]. In this paper we will use Bernstein-Chlodowsky polynomials on $\Omega$ which is introduced in [7]. Let $f \in L^2_p(\Omega)$,

$$B_n(f; x, y) = \sum_{k=0}^{n} C_n^k \left(1 - \frac{x + y}{b_n}\right)^{n-k} \sum_{i=0}^{k} f\left(\frac{k-i}{n} b_n, \frac{i}{n} b_n\right) C_i^k \left(\frac{x}{b_n}\right)^{k-i} \left(\frac{y}{b_n}\right)^i$$

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for \((x,y) \in \Delta_{b_n}\). We note that formula (1) is the sequence of linear positive operators in the space of integrable functions \(L_p\) of two variables, that is these linear positive operators translate a positive function to another positive one. But, in general, the function is not necessarily a continuous one in \(L_p\) space. We can not use Korovkin’s Theorem. First, We give certain results which are necessary to prove the main results.

**Lemma 1.** Suppose that \(e_{k,m}(t_s) = t^k\) then

\[
\begin{align*}
B_n(e_{0,0};x,y) &= 1 \\
B_n(e_{1,0};x,y) &= x \\
B_n(e_{0,1};x,y) &= y \\
B_n(e_{2,0};x,y) &= x^2 + \frac{x(b_n - x)}{n} \\
B_n(e_{0,2};x,y) &= y^2 + \frac{y(b_n - y)}{n}
\end{align*}
\]

Simple calculations can be calculated above Lemma.

**Theorem 1 ([7]).** Let \(f \in L_p(\Omega)\) and \(a\) be a fixed point in \((0, b_n)\). If, for every \((x,y) \in \Delta_a\) and \((t,s) \in \Delta_{b_n}\)

\[
\frac{|f(t,s) - f(x,y)|}{|(t,s) - (x,y)|} \leq M
\]

(2)

hold with the constant \(M\), then

\[
\|B_n(f) - f\|_{L_p(\Delta_a)} \to 0, n \to \infty.
\]

Where \(a > 0\).

**2. Main Theorems**

To simplify notation, we need the following.

\[
L^2_p(\Omega_1) = \{f \in L_p(\Omega_1) : \Delta |^a f \in L_p(\Delta_a), |a| = 2, \Omega_1 \subset [0, b_n] \times [0, b_n]\}.
\]

We consider also the following K-functional of Peetre;

\[
K_p(f; \delta) = \inf_{g \in L^2_p(\Delta_a)} \{\|f - g\|_{L_p(\Delta_a)} + \delta(\|g\|_{L^2_p(\Delta_a)})\}, \delta \geq 0.
\]

for \(f \in L_p(\Delta_a)\), we have \(\lim_{\delta \to 0} K(f; \delta) = 0\). Therefore the K-functional gives the degree of approximation of a function \(f \in L_p(\Delta_a)\) by smoother functions \(g \in L^2_p(\Delta_a)\). Remember that the second order integral modulus of smoothness is given by

\[
\omega_{2,p}(f; \delta) = \sup_{0 \leq h \leq \delta} \|f(x + h) - 2f(x) + f(x - h)\|_{L_p(\Delta_a)}(I_h)
\]
for an $f \in L_p(\Delta_a)$, where $I_h$ indicates that the $L_p$-norm is taken over the interval $[h, b_n - h]$. It is also known that there are constants $a_1 > 0, a_2 > 0$, independent of $f$ and $p$ such that

$$a_1 \omega_{2,p}(f ; \delta^{1/2}) \leq K_p(f ; \delta) \leq \min(1, \delta)\|f\|_{L_p(\Delta_a)} + 2a_2 \omega_{2,p}(f; \delta^{1/2})$$  \hspace{1cm} (3)

We prove the following theorems:

**Theorem 2.** Let $f \in L^2_p(\Omega_1), 1 \leq p < \infty$ and $a$, $M$ are constants, If the condition,

$$\frac{|f(t,s) - f(x,y)|}{|(t,s) - (x,y)|} \leq M, t \in (a, b_n], s \in (a, b_n], (x,y) \in \Delta_a$$

is satisfied, then

$$\|B_n(f) - f\|_{L^2_p(\Delta_a)} \leq C_p(\|f\|_{L^2_p(\Delta_a)})\delta_n, \delta_n = \frac{a(b_n + a)}{n}$$

$$C_p = \left\{ \begin{array}{ll} \frac{p}{p+1} & p > 1, \\ 1 + a^2 & p = 1. \end{array} \right.$$  \hspace{1cm} (5)

**Proof.** For $f \in L^2_p(\Omega_1)$ we can write that,

$$B_n(f(t,s) - f(x,y); x, y) = f_x(x,y)B_n((t-x); x, y) + f_y(x,y)B_n((s-y); x, y)$$

$$+ B_n\left( \int_x^t f_{uu}(u,y)(u-t)du; x, y \right)$$

$$+ B_n\left( \int_y^s f_{kk}(x,k)(k-s)dk; x, y \right)$$

$$+ B_n\left( \int_x^t \int_y^s f_{ts}(t,s)dsdt; x, y \right)$$

Now, we need the Hardy-Littlewood majorante of $f_{xx}$ at $x$. Which is defined as following:

$$\varphi_{f_{xx}(x,y)} = \sup_{0 \leq t \leq x, t \neq x} \left( \frac{1}{t - x} \right) \int_x^t f_{uu}(u,y)du$$

and using following inequality

$$\int_\Omega |\varphi_{f_{xy}(x,y)}|^p dxdy \leq 2\left( \frac{p}{p+1} \right)^p \int_0^a \int_0^a |f_{ts}(t,s)|^p dsdt$$

using $L_p$-norm, we get

$$|B_1(x,y)| + |B_2(x,y)| + |B_3(x,y)| \leq \varphi_{f_{xx}(x,y)}\delta_n + \varphi_{f_{yy}(x,y)}\delta_n + \varphi_{f_{xy}(x,y)}\sqrt{\delta_n}$$
Theorem 3. Let $f \in L^2_p(\Omega^1)$, $1 \leq p < \infty$ and $f$ satisfies the condition (2) then the following inequality

$$
\|B_n f - f\|_{L^p_p(\Delta_a)} \leq M_p[\|f\|_{L^2_p(\Delta_a)} \delta_n + \omega_{2,p}(f; \delta^{1/2})]
$$

holds. Where $a$, $M$ are constants.

Proof. For all sufficiently large $n$, from Theorem 2 we can write

$$
\|B_n h - h\|_{L^p_p(\Delta_a)} \leq \cases{\| \varepsilon \|_{L^p_p(\Delta_a)}, & h \in L^p_p(\Delta_a), \\
C_p\|f\|_{L^p_p(\Delta_a)} \delta_n, & h \in L^2_p(\Delta_a)
}
$$

where $C_p$ is positive constant which independent of $h, n$ and where $h$ satisfies (2). When $f \in L^2_p(\Omega^1)$ and $g \in L^2_p(\Delta_a)$ the condition (2) is satisfied then

$$
\|B_n f - f\|_{L^p_p(\Delta_a)} \leq \|B_n(f - g)\|_{L^p_p(\Delta_a)} + \|B_n g - g\|_{L^p_p(\Delta_a)}
$$
\[
\|f - g\|_{L_p(\Delta_a)} \leq (\varepsilon + M\delta_n a)\|f - g\|_{L_p(\Delta_a)} + C_p \|g\|_{L_p^2(\Delta_a)}\delta_n
\]

where \( \widetilde{M} = \max\{\varepsilon + M\delta_n a, C_p\} \).

Using the K-functional we get,

\[
\|B_n f - f\|_{L_p(\Delta_a)} \leq \widetilde{M} \sup_{g \in L_p^2(\Delta_a)} [\|f - g\|_{L_p(\Delta_a)} + \|g\|_{L_p^2(\Delta_a)} \delta_n]
\]

since, for a sufficiently large \( n, \delta_n \) and from (3),

\[
K_p(f; \delta) \leq \delta_n \|f\|_{L_p(\Delta_a)} + 2a_1 \omega_{2,p}(f; \delta^{(1/2)})
\]

\[
\widetilde{M}K_p(f; \delta) \leq \widetilde{M}[\delta_n \|f\|_{L_p(\Delta_a)} + 2a_1 \omega_{2,p}(f; \delta^{(1/2)})]
\]

we obtain (4),

\[
\|B_n f - f\|_{L_p(\Delta_a)} \leq M_p [\|f\|_{L_p^2(\Delta_a)} \delta_n + \omega_{2,p}(f; \delta^{(1/2)})].
\]

Thus, the proof is completed.

References


