On Köthe-Toeplitz and Null Duals of Some Difference Sequence Spaces Defined by Orlicz Functions

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Abstract. The main aim of this paper is to compute Köthe-Toeplitz and Null duals of some difference sequence spaces, defined by means of a fixed sequence of multiplier and by an Orlicz function. Further the coincidence for three pairs of analogous spaces is established.

2000 Mathematics Subject Classifications: 40A05, 40C05, 46A45.

Key Words and Phrases: Difference sequence spaces, Orlicz function, Köthe-Toeplitz dual, Null dual.

1. Introduction and Preliminaries

Throughout this section $w$, $\ell_\infty$, $\ell_1$, $c$ and $c_0$ denote the spaces of all, bounded, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms respectively.

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An Orlicz function is a function $M : [0, \infty) \rightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$.

An Orlicz function $M$ is said to satisfy the $\Delta_2$-condition for all values of $u$, if there exists a constant $K > 0$, such that

$$M(2u) \leq KM(u) \quad (u \geq 0).$$

The above $\Delta_2$-condition implies $M(lu) \leq K^{\log_2 K}M(u)$, for all $u > 0$, $l > 1$.

For details on integral representation of Orlicz function as well as on complementary Orlicz functions one may refer to [7, 12].

For an Orlicz function $M$, we have the following inequality:

$$M(\lambda x) < \lambda M(x),$$

for all $x \geq 0$ and $\lambda$ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [9] used the Orlicz function and introduced the sequence space $\ell_M$ as follows:

$$\ell_M = \{(x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{, for some } \rho > 0\}.$$

They proved that $\ell_M$ is a Banach space normed by

$$\|(x_k)\| = \inf \{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}.$$

Let $\Lambda = (\lambda_k)$ be a sequence of non-zero scalars. Then for $E$ a sequence space, the multiplier sequence space $E(\Lambda)$, associated with the multiplier sequence $\Lambda$ is defined as

$$E(\Lambda) = \{(x_k) \in w : (\lambda_k x_k) \in E\}.$$

The scope for the studies on sequence spaces was extended by using the notion of associated multiplier sequences. Goes and Goes [4] defined the differentiated
sequence space $dE$ and integrated sequence space $\int E$ for a given sequence space $E$, using the multiplier sequences $(k^{-1})$ and $(k)$ respectively. A multiplier sequence can be used to accelerate the convergence of the sequences in some spaces. In some sense, it can be viewed as a catalyst, which is used to accelerate the process of chemical reaction.

The notion of difference sequence space was introduced by Kizmaz [6], who studied the difference sequence spaces $Z(\Delta)$, for $Z = \ell_\infty, c, c_0$ and defined as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.

In this paper our aim is to investigate some important structures of some spaces which are defined using an Orlicz function and a multiplier sequence. These spaces generalize the spaces $Z(\Delta)$, for $Z = \ell_\infty, c, c_0$ introduced and studied by Kizmaz [6].

Let $\Lambda = (\lambda_k)$ be a non-zero sequence of scalars. Then we define the following sequence spaces for an Orlicz function $M$:

$$c_0(M, \Lambda, \Delta) = \{x = (x_k) : \lim_k M\left(\frac{|\Delta \lambda_k x_k|}{\rho}\right) = 0, \text{ for some } \rho > 0\},$$

$$c(M, \Lambda, \Delta) = \{x = (x_k) : \lim_k M\left(\frac{|\Delta \lambda_k x_k - L|}{\rho}\right) = 0, \text{ for some } L \text{ and } \rho > 0\},$$

$$\ell_\infty(M, \Lambda, \Delta) = \{x = (x_k) : \sup_k M\left(\frac{|\Delta \lambda_k x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\},$$

where $\Delta \lambda_k x_k = \lambda_k x_k - \lambda_{k+1} x_{k+1}$, for all $k \in \mathbb{N}$.

It is obvious that $c_0(M, \Lambda, \Delta) \subset c(M, \Lambda, \Delta) \subset \ell_\infty(M, \Lambda, \Delta)$.

Throughout the paper $X$ will denote one of the sequence spaces $c_0, c$ and $\ell_\infty$. The sequence spaces $X(M, \Lambda, \Delta)$ are Banach spaces normed by

$$\|x\|_{\Delta} = |\lambda_1 x_1| + \inf\{\rho > 0 : \sup_k M\left(\frac{|\Delta \lambda_k x_k|}{\rho}\right) \leq 1\}.$$
if and only if \((\Delta^{-1} \lambda_k x_k) \in X(M)\). Now for \(x \in X(M, \Lambda, \Delta^{-1})\), we define

\[
\|x\|_{\Delta^{-1}} = \inf\{\rho > 0 : \sup_k \left(\frac{|\Delta^{-1} \lambda_k x_k|}{\rho}\right) \leq 1\}.
\]

It can be shown that \(X(M, \Lambda, \Delta)\) is a BK-space under the norms \(\|\|_\Delta\) and \(\|\|_{\Delta^{-1}}\) respectively and it is obvious that the norms \(\|\|_\Delta\) and \(\|\|_{\Delta^{-1}}\) are equivalent.

Obviously \(\Delta^{-1} : X(M, \Lambda, \Delta^{-1}) \to X(M)\), defined by \(\Delta^{-1} x = y = (\Delta^{-1} \lambda_k x_k)\), is isometric isomorphism.

Hence \(c_0(M, \Lambda, \Delta^{-1})\), \(c(M, \Lambda, \Delta^{-1})\) and \(\ell_\infty(M, \Lambda, \Delta^{-1})\) are isometrically isomorphic to \(c_0(M)\), \(c(M)\) and \(\ell_\infty(M)\) respectively. From abstract point of view \(X(M, \Lambda, \Delta^{-1})\) is identical with \(X(M)\), for \(X = c_0, c\) and \(\ell_\infty\).

The results obtained in the next section also hold for the spaces \(c_0(M, \Lambda, \Delta^{-1})\), \(c(M, \Lambda, \Delta^{-1})\) and \(\ell_\infty(M, \Lambda, \Delta^{-1})\) as well as for the spaces associated with these three spaces.

Now we define the spaces \(\tilde{c}_0(M, \Lambda, \Delta)\), \(\tilde{c}(M, \Lambda, \Delta)\) and \(\tilde{\ell}_\infty(M, \Lambda, \Delta)\) as follows:

\(\tilde{c}_0(M, \Lambda, \Delta)\) is a subspace of \(c_0(M, \Lambda, \Delta)\) consisting of those \(x \in c_0(M, \Lambda, \Delta)\) such that

\[
\lim_k M\left(\frac{\Delta \lambda_k x_k}{d}\right) = 0 \text{ for each } d > 0.
\]

Similarly we can define \(\tilde{c}(M, \Lambda, \Delta)\) and \(\tilde{\ell}_\infty(M, \Lambda, \Delta)\) as subspace of \(c(M, \Lambda, \Delta)\) and \(\ell_\infty(M, \Lambda, \Delta)\) respectively.

It is obvious that \(\tilde{c}(M, \Lambda, \Delta) \subset \tilde{c}(M, \Lambda, \Delta) \subset \tilde{\ell}_\infty(M, \Lambda, \Delta)\). Also as above we can show that \(\tilde{c}_0(M, \Lambda, \Delta)\), \(\tilde{c}(M, \Lambda, \Delta)\) and \(\tilde{\ell}_\infty(M, \Lambda, \Delta)\) are isometrically isomorphic to \(\tilde{c}_0(M)\), \(\tilde{c}(M)\) and \(\tilde{\ell}_\infty(M)\) respectively.

Moreover \(X(M, \Lambda) \subset X(M, \Lambda, \Delta)\) and \(\tilde{X}(M, \Lambda) \subset \tilde{X}(M, \Lambda, \Delta)\) which can be shown by using the following inequality:

\[
M\left(\frac{\Delta \lambda_k x_k}{2\rho}\right) \leq \frac{1}{2} M\left(\frac{\lambda_k x_k}{\rho}\right) + \frac{1}{2} M\left(\frac{\lambda_{k+1} x_{k+1}}{\rho}\right).
\]
2. Köthe-Toeplitz and Null Dual Spaces

In this section we compute Köthe-Toeplitz or $\alpha$-dual and Null or $N$-dual of some difference sequence spaces as described in the preceding section.

Let $E$ and $F$ be two sequence spaces. Then the $F$ dual of $E$ is defined as

$$E^F = \{ (x_k) \in w : (x_k y_k) \in F \text{ for all } (y_k) \in E \}.$$

For $F = \ell_1$ and $c_0$, the duals are termed as $\alpha$-(or Köthe-Toeplitz) dual and $N$-(or Null) dual of $E$ and denoted by $E^\alpha$ and $E^N$ respectively. If $X \subset Y$, then $Y^z \subset X^z$ for $z = \alpha, N$.

**Lemma 1.** $x \in \ell_\infty(M, \Lambda, \Delta)$ implies $\sup_k M(\frac{\lambda_k x_k - \lambda_{k+1} x_{k+1}}{\rho}) < \infty$, for some $\rho > 0$.

**Proof.** Let $x \in \ell_\infty(M, \Lambda, \Delta)$, then

$$\sup_k M(\frac{\lambda_k x_k - \lambda_{k+1} x_{k+1}}{\rho}) < \infty,$$

for some $\rho > 0$.

Then there exists a $U > 0$ such that

$$M(\frac{\lambda_k x_k - \lambda_{k+1} x_{k+1}}{\rho}) < U, \text{ for all } k \in \mathbb{N}.$$

Taking $\eta = k \rho$, for an arbitrary fixed positive integer $k$, by the subadditivity of modulus, the monotonicity and convexity of $M$:

$$M(\frac{\lambda_1 x_1 - \lambda_{k+1} x_{k+1}}{\eta}) < \frac{1}{k} \sum_{i=1}^{k} M(\frac{\lambda_i x_i - \lambda_{i+1} x_{i+1}}{\rho}) < U.$$

Then the above inequality, the inequality

$$\frac{\lambda_{k+1} x_{k+1}}{(k+1)\rho} \leq \frac{1}{k+1}(\frac{\lambda_1 x_1}{\rho} + \frac{k\lambda_1 x_1 - \lambda_{k+1} x_{k+1}}{k\rho})$$

and the convexity of $M$ imply

$$M(\frac{\lambda_{k+1} x_{k+1}}{(k+1)\rho}) \leq \frac{1}{k+1}(M(\frac{\lambda_1 x_1}{\rho}) + kM(\frac{\lambda_1 x_1 - \lambda_{k+1} x_{k+1}}{k\rho})).$$
\[
\leq \max\{M(\frac{\lambda_1 x_1}{\rho}), U\} < \infty
\]

Hence we have the desired result.

**Lemma 2.** \( x \in \ell_{\infty}(M, \Lambda, \Delta) \) implies \( \sup_k k^{-1} |\lambda_kx_k| < \infty \).

**Proof.** Proof is obvious by using Lemma 1.

**Remark 1.** Similar results as in Lemma 1 and Lemma 2 hold for \( \tilde{\ell}_{\infty}(M, \Lambda, \Delta) \) also, where the statement 'for some \( \rho > 0 \)' should be replaced by 'for every \( \rho > 0 \)'.

For the next theorem, let \( D_1 = \{a = (a_k) : \sum_{k=1}^{\infty} k|\lambda_k^{-1}a_k| < \infty\} \), \( D_2 = \{b = (b_k) : \sup_k k^{-1}|\lambda_kb_k| < \infty\} \).

**Theorem 1.** Let \( M \) be an Orlicz function. Then

(i) \( [c(M, \Lambda, \Delta)]^\alpha = [\ell_{\infty}(M, \Lambda, \Delta)]^\alpha = D_1 \),

(ii) \( [\tilde{c}(M, \Lambda, \Delta)]^\alpha = [\tilde{\ell}_{\infty}(M, \Lambda, \Delta)]^\alpha = D_1 \),

(iii) \( D_1^\alpha = D_2 \).

**Proof.** (i) Let \( a \in D_1 \), then \( \sum_{k=1}^{\infty} k|\lambda_k^{-1}a_k| < \infty \). Now for any \( x \in \ell_{\infty}(M, \Lambda, \Delta) \) we have \( \sup_k k^{-1}|\lambda_kx_k| < \infty \). Then we have

\[
\sum_{k=1}^{\infty} k|a_kx_k| \leq \sup_k k^{-1} |\lambda_kx_k| \sum_{k=1}^{\infty} |k\lambda_k^{-1}a_k| < \infty.
\]

Hence \( a \in [\ell_{\infty}(M, \Lambda, \Delta)]^\alpha \).

Thus

\[
D_1 \subseteq [\ell_{\infty}(M, \Lambda, \Delta)]^\alpha \tag{1}
\]

Again we know

\[
[\ell_{\infty}(M, \Lambda, \Delta)]^\alpha \subseteq [c(M, \Lambda, \Delta)]^\alpha \subseteq [c_0(M, \Lambda, \Delta)]^\alpha \tag{2}
\]
Conversely suppose that $a \in [c(M, \Lambda, \Delta)]^\alpha$. Then $\sum_{k=1}^{\infty} |a_kx_k| < \infty$, for each $x \in c(M, \Lambda, \Delta)$. So we take

$$x_k = \lambda_k^{-1}k, k \geq 1$$

then

$$\sum_{k=1}^{\infty} |k\lambda_k^{-1}a_k| = \sum_{k=1}^{\infty} |a_kx_k| < \infty.$$ 

This implies that $a \in D_1$. Thus

$$[c(M, \Lambda, \Delta)]^\alpha \subseteq D_1.$$ 

Combining (3) with (1), (2) it follows

$$[c(M, \Lambda, \Delta)]^\alpha = [\ell_\infty(M, \Lambda, \Delta)]^\alpha = D_1$$

This completes the proof of part (i).

(ii) Proof is similar to that of part (i).

(iii) The proof of the inclusion $D_1^\alpha \supseteq D_2$ is similar to that of $D_1 \subseteq [\ell_\infty(M, \Lambda, \Delta)]^\alpha$.

For the converse part suppose $a \in D_1^\alpha$ and $a \notin D_2$. Then we have

$$\sup_k |k^{-1}\lambda_k a_k| = \infty$$

Hence we can find a strictly increasing sequence $(k_j)$ of positive integers $k_j$ such that

$$|k_j^{-1}\lambda_k a_k| > j^2 \text{ for all } j \geq 1$$

We define the sequence $x$ by

$$x_k = \begin{cases} 
|a_{k_j}^{-1}|, & \text{if } k = k_j \\
0, & \text{otherwise}
\end{cases}$$
Then $x \in D_1$, because
\[
\sum_{k=1}^{\infty} |k\lambda_k^{-1}x_k| = \sum_{j=1}^{\infty} |k_j\lambda_j^{-1}a_k| \leq \sum_{j=1}^{\infty} j^{-2} < \infty
\]
Thus $x \in D_1$ but $\sum_{k=1}^{\infty} |a_kx_k| = \sum_{j=1}^{\infty} |a_jx_k| = \infty$. This is a contradiction to $a \in D_1^\alpha$.
Hence $a \in D_2$. This completes the proof.

If we take $\lambda_k = 1$, for all $k \in \mathbb{N}$ in Theorem 1, then we obtain the following corollary.

**Corollary 1.** For $X = c$ and $\ell_\infty$,
\begin{enumerate}[(i)]  
  
  (i) $[X(M, \Delta)]^\alpha = [\tilde{X}(M, \Delta)]^\alpha = H_1$,
  
  (ii) $H_1^\alpha = H_2$,
\end{enumerate}
where
\[
H_1 = \{a = (a_k) : \sum_{k=1}^{\infty} |ka_k| < \infty\}
\]
and
\[
H_2 = \{b = (b_k) : \sup_k |k^{-1}b_k| < \infty\}.
\]

For the next theorem, let $G_1 = \{a = (a_k) : \lim_k k\lambda_k^{-1}a_k = 0\}$.

**Theorem 2.** Let $M$ be an Orlicz function. Then
\begin{enumerate}[(i)]  
  
  (i) $[c(M, \Lambda, \Delta)]^N = [\ell_\infty(M, \Lambda, \Delta)]^N = G_1$,
  
  (ii) $[\tilde{c}(M, \Lambda, \Delta)]^N = [\ell_\infty(M, \Lambda, \Delta)]^N = G_1$.
\end{enumerate}

**Proof.** (i) Proof is immediate using Lemma 2.

(ii) Proof is similar to that of part (i).

If we take $\lambda_k = 1$, for all $k \in \mathbb{N}$ in Theorem 2, then we obtain the following corollary.
Corollary 2. For $X = c$ and $\ell_\infty$,

(i) $[X(M, \Delta)]^N = [\tilde{X}(M, \Delta)]^N = L_1$,

where $L_1 = \{ a = (a_k) : \lim_k k a_k = 0 \}$.

Theorem 3. If $M$ satisfies the $\Delta_2$-condition, then we have $X(M, \Lambda, \Delta) = \tilde{X}(M, \Lambda, \Delta)$, for every $X = c_0, c$ and $\ell_\infty$.

Proof. We give the proof for $X = \ell_\infty$ and for other spaces it will follow on applying similar arguments.

To prove the theorem, it is enough to show that $\ell_\infty(M, \Lambda, \Delta)$ is a subspace of $\tilde{\ell}_\infty(M, \Lambda, \Delta)$.

Let $x \in \ell_\infty(M, \Lambda, \Delta)$, then for some $\rho > 0$,

$$\sup_k M(\frac{\Delta \lambda_k x_k}{\rho}) < \infty$$

Therefore

$$M(\frac{\Delta \lambda_k x_k}{\rho}) < \infty, \text{ for every } k \in \mathbb{N}.$$ 

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$ then $M(\frac{\Delta \lambda_k x_k}{\eta}) < \infty$ for every $k \in \mathbb{N}$. Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$.

Since $M$ satisfies the $\Delta_2$-condition, there exists a constant $K$ such that

$$M(\frac{\Delta \lambda_k x_k}{\eta}) \leq K(\frac{\rho}{\eta}) \log_2^K M(\frac{\Delta \lambda_k x_k}{\rho}) < \infty \text{ for every } k \in \mathbb{N}.$$ 

Now let us denote

$$S = \sup_k M(\frac{\Delta \lambda_k x_k}{\rho}) < \infty, \text{ for the fixed } \rho > 0.$$ 

Then it follows that for every $\eta > 0$, we have

$$\sup_k M(\frac{\Delta \lambda_k x_k}{\eta}) \leq K(\frac{\rho}{\eta}) \log_2^K S < \infty.$$
ACKNOWLEDGEMENTS  The author is very grateful to the anonymous referee for the constructive comments and helpful suggestions which have improved the presentation of this paper.

References