Module Extension Banach Algebras and \((\sigma, \tau)\)-amenability

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Abstract. In this paper among other things we find some necessary and sufficient conditions for a Banach algebra \(\mathcal{A}\), to be \((\sigma, \tau)\)-amenable, where \(\sigma\) and \(\tau\) are continuous homomorphisms on \(\mathcal{A}\).

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1. Introduction.

Let \(\mathcal{A}\) be a Banach algebra and \(\mathcal{X}\) be a Banach \(\mathcal{A}\)-bimodule, that \(\mathcal{X}\) is both a Banach space and an algebraic \(\mathcal{A}\)-bimodule, and the module operations \((a, x) \mapsto ax\) and \((a, x) \mapsto xa\) from \(\mathcal{A} \times \mathcal{X}\) into \(\mathcal{X}\) are (jointly) continuous. Then \(\mathcal{X}^*\) is also a Banach \(\mathcal{A}\)-bimodule under the following module actions:

\[
(a \cdot f)(x) = f(xa),
\]

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\( (f \cdot a)(x) = f(ax), \)

\( a \in \mathcal{A}, x \in \mathcal{X}, f \in \mathcal{X}^*. \)

Let \( \mathcal{A} \) be a Banach algebra. Given \( f \in \mathcal{A}^* \) and \( F \in \mathcal{A}^{**} \), then \( Ff \) and \( fF \) are defined in \( \mathcal{A}^* \) by the following formulae

\[
Ff(a) = F(f \cdot a), \quad fF(a) = F(a \cdot f) \quad (a \in \mathcal{A}).
\]

Next, for \( F, G \in \mathcal{A}^{**} \), \( FG \) is defined in \( \mathcal{A}^{**} \) by the formulae

\[
(FG)(f) = F(Gf),
\]

this product is called first Arens product on \( \mathcal{A}^{**} \) and \( \mathcal{A}^{**} \) with the first Arens product is a Banach algebra.

Let \( \mathcal{A} \) be a Banach algebra and \( \mathcal{X} \) be a Banach \( \mathcal{A} \)-bimodule. The Banach space \( \mathcal{X}^{**} \) is a Banach \( \mathcal{A}^{**} \)-bimodule under following actions

\[
F \cdot G = w^* - \lim_{i,j} a_i x_j, \quad G \cdot F = w^* - \lim_{i,j} x_j a_i
\]

where \( F = w^* - \lim_i a_i, \ G = w^* - \lim_j x_j, \ (a_i) \) is a net in \( \mathcal{A}, \ (x_j) \) and is a net in \( \mathcal{X} \).

Suppose that \( \varphi : \mathcal{A} \to \mathcal{B} \) is a Banach algebra homomorphism. The Banach algebra \( \mathcal{B} \) is considered as a Banach \( \mathcal{A} \)-bimodule by the following module actions

\[
a \cdot b = \varphi(a)b, \quad b \cdot a = b \varphi(a) \quad (a \in \mathcal{A}, b \in \mathcal{B})
\]

we denote \( \mathcal{B}_{\mathcal{A}} \) the above \( \mathcal{A} \)-bimodule.

Let \( \mathcal{A} \) be a Banach algebra and \( \sigma, \tau \) be continuous homomorphisms on \( \mathcal{A} \). Suppose that \( \mathcal{X} \) is a Banach \( \mathcal{A} \)-bimodule. A linear mapping \( d : \mathcal{A} \to \mathcal{X} \) is called a \((\sigma, \tau)\)-derivation if

\[
d(ab) = d(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in \mathcal{A}).
\]

For example every ordinary derivation of an algebra \( \mathcal{A} \) into an \( \mathcal{A} \)-bimodule \( \mathcal{X} \) is an \((id_{\mathcal{A}}, id_{\mathcal{A}})\)-derivation, where \( id_{\mathcal{A}} \) is the identity mapping on the algebra \( \mathcal{A} \).
A linear mapping \( d : \mathcal{A} \rightarrow \mathcal{X} \) is called \((\sigma, \tau)\)-inner derivation if there exists \( x \in \mathcal{X} \) such that \( d(a) = \tau(a)x - x\sigma(a) \quad (a \in \mathcal{A}) \). See also [3–6].

We denote the set of continuous \((\sigma, \tau)\)-derivations from \( \mathcal{A} \) into \( \mathcal{X} \) by \( Z^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}) \) and the set of inner \((\sigma, \tau)\)-derivations by \( B^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}) \). We define the space \( H^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}) \) as the quotient space \( Z^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X})/B^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}) \). The space \( H^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}) \) is called the first \((\sigma, \tau)\)-cohomology group of \( \mathcal{A} \) with coefficients in \( \mathcal{X} \).

\( \mathcal{A} \) is called \((\sigma, \tau)\)-amenable if \( H^1_{(\sigma, \tau)}(\mathcal{A}, \mathcal{X}^*) = \{0\} \), for each Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \).

Let \( \mathcal{A} \) be a Banach algebra and let \( \mathcal{X} \) be a Banach \( \mathcal{A} \)-bimodule. Define \( \mathcal{A} \oplus_1 \mathcal{X} \) by actions:

\[
(a, x) + (b, y) = (a + b, x + y)
\]

\( a(b, x) = (ab, ax) \), \( (b, x)a = (ba, xa) \)

\[
(a, x)(b, y) = (ab, ay + xb),
\]

for every \( a, b \in \mathcal{A} \) and \( x, y \in \mathcal{X} \).

It is clear \( \mathcal{A} \oplus_1 \mathcal{X} \) is a Banach algebra with the following norm:

\[
\|(a, x)\| = \|a\| + \|x\|.
\]

This Banach algebra is called module extension Banach algebra.

We use some ideas and terminology of [2] to investigate \((\sigma, \tau)\)-amenability of Banach algebras.

2. \((\sigma, \tau)\)-amenability of Banach Algebras.

Let \( \mathcal{A} \) be a Banach algebra and let \( \sigma, \tau \) be continuous homomorphisms on \( \mathcal{A} \). Suppose that \( \mathcal{X} \) is a Banach \( \mathcal{A} \)-bimodule. Then \( \mathcal{X} \) is a Banach \( \mathcal{A} \)-bimodule by the following module actions:

\[
a \cdot x = \tau(a)b, \quad x \cdot a = b\sigma(a) \quad (a \in \mathcal{A}, x \in \mathcal{X}).
\]
We denote $\mathcal{A}_{(\sigma, \tau)}$ for this $\mathcal{A}$-bimodule. It is easy to check that $(\mathcal{A}_{(\sigma, \tau)})^* = X^*_{(\tau, \sigma)}$, and that every $(\sigma, \tau)$-derivation from $\mathcal{A}$ into $\mathcal{X}$ is a derivation from $\mathcal{A}$ into $\mathcal{X}_{(\sigma, \tau)}$. Thus we can show that $\mathcal{A}$ is amenable, if and only if $\mathcal{A}$ is $(\sigma, \tau)$-amenable, for each $\sigma, \tau \in \text{Hom}(\mathcal{A})$. First we give the following examples for $(\sigma, \tau)$-amenability of Banach algebras.

**Example 2.1.** It is easy to see that $\ell^1$ is a Banach algebra equipped with the following product 

\[
a \cdot b = a(1)b \quad (a, b \in \ell^1),
\]

and $\ell^1$ has a left identity $e$ defined by

\[
e(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n \neq 1.
\end{cases}
\]

The dual space $(\ell^1)^* = \ell^\infty$ is a $\ell^1$-bimodule via the ordinary actions as follows

\[
a \cdot f = f(a)e, \quad f \cdot a = a(1)f \quad (a \in \ell^1, f \in \ell^\infty),
\]

where $e$ is regarded as an element of $\ell^\infty$.

Next let $\sigma : \ell^1 \to \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $\sigma(b)(a(1) - \sigma(a)(1)) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we have

\[
(\sigma(a))(1) = a(1) \quad (a \in \ell^1) \quad (2.1)
\]

In [5] has been shown that $\ell^1$ is $(\sigma, \tau)$-weakly amenable for all homomorphisms $\sigma, \tau$ but for some homomorphisms $\sigma$ and $\tau$ it is not $(\sigma, \tau)$-amenable. In the following we prove if the Banach algebra $\ell^1$ is $(\sigma, \tau)$-amenable, then $\tau(a) = a(1)c$ where $c(1) = 1$. 


Let $\mathcal{B} = \ell^1$ by product $a \cdot b = a(2)b$. Then $\mathcal{B}$ is a Banach algebra and for each bounded homomorphism $\psi : \mathcal{B} \to \mathcal{B}$ we have $\left(\psi(a)\right)(2) = a(2)$. Let $a \in \ell^1$ define $a' \in \ell^1$ by $a' = (a(2), a(1), a(3), \cdots)$. Let $\varphi : \ell^1 \to \mathcal{B}$ defined by $\varphi(a) = a'$. It is clear that $\varphi$ is a homomorphism. Consider the Banach $\ell^1$-bimodule $\mathcal{B}_\varphi$ under actions $a \circ b = \varphi(a) \cdot b = a' \cdot b = a'(2)b = a(1)b$ and $b \circ a = b \cdot \varphi(a) = b \cdot a' = b(2)a'$ for each $a \in \ell^1$, $b \in \mathcal{B}_\varphi$. Let $D : \ell^1 \to \mathcal{B}_\varphi$ be a bounded $(\sigma, \tau)$-derivation. We have

\[
\begin{align*}
(D(a \cdot b))(c) &= D(a)\sigma(b)(c) + \tau(a)D(b)(c) \\
(a(1)D(b))(c) &= D(a)(\sigma(b) \circ c) + D(b)(c \circ \tau(a)) \\
(a(1)D(b))(c) &= b(1)D(a)(c) + c(2)D(b)(\tau(a))
\end{align*}
\]

for all $a, b \in \ell^1$ and $c \in \mathcal{B}_\varphi$.

By taking $a = b$ we obtain $D(a)(\tau(a)) = 0$. Also by taking $c \in \mathcal{B}_\varphi$ such that $c(2) = 0$ we can conclude $a(1)D(b) = b(1)D(a)$.

If $\ell^1$ is $(\sigma, \tau)$-amenable, then there exists $f \in B_\varphi^*$ such that $D = D_f$ is a $(\sigma, \tau)$-inner derivation. So we have

\[
\begin{align*}
(a(1)D_f)(b) &= b(1)D_f(a) \\
(a(1)f(b(1)c - c(2)\tau(b))) &= b(1)f(a(1)c - c(2)\tau(a))
\end{align*}
\]

for all $a, b \in \ell^1$ and $c \in \mathcal{B}_\varphi$.

Then $f(b(1)c(2)\tau(a) - a(1)c(2)\tau(b)) = 0$. Since $f \in B_\varphi^*$ is arbitrary, immediately is conclude $a(1)\tau(b) = b(1)\tau(a)$. By taking $b = e$ we have $\tau(a) = a(1)\tau(e)$, where $\tau(e)(1) = 1$.

So we have the following result.

**Corollary 2.1.** Let $\sigma, \tau$ be two continuous homomorphisms on $\ell^1$ (by above product). If $\ell^1$ is $(\sigma, \tau)$-amenable then there is $c \in \ell^1$ such that $\tau(a) = a(1)c$, and $c(1) = 1$. 
Example 2.2. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ has a bounded approximate identity if and only if $\mathcal{A}$ is $(\text{id}, 0)$ and $(0, \text{id})$-amenable.

Corollary 2.2. Let $\mathcal{A}$ be a $C^*$-algebra or $\mathcal{A} = L^1(G)$ for a locally compact topological group $G$. Then $\mathcal{A}$ is $(\text{id}, 0)$ and $(0, \text{id})$-amenable.

Let $T : \mathcal{A} \to \mathcal{B}$ be a continuous linear map between Banach algebras. Two continuous linear maps $T' : \mathcal{B}^* \to \mathcal{A}^*$ and $T'' : \mathcal{A}^{**} \to \mathcal{B}^{**}$ are known, that are defined by the following formula

$$
(T'(f))(a) = f(T(a)), \quad (T''(G))(f) = G(T'(f))
$$

where $a \in \mathcal{A}$, $f \in \mathcal{B}^*$ and $G \in \mathcal{A}^{**}$.

Lemma 2.1. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and let $\sigma$ and $\tau$ be two continuous homomorphisms on $\mathcal{A}$. Suppose that $D : \mathcal{A} \to \mathcal{X}$ is $(\sigma, \tau)$-derivation. Then $D'' : \mathcal{A}^{**} \to \mathcal{X}^{**}$ is a $(\sigma'', \tau'')$-derivation.

Proof. Let $F, G \in \mathcal{A}^{**}$ and let $F = w^* - \lim_\alpha a_\alpha, G = w^* - \lim_\beta b_\beta$ in $\mathcal{A}^{**}$, where $(a_\alpha), (b_\beta)$ are nets in $\mathcal{A}$ with $||a_\alpha|| \leq ||F||, ||b_\beta|| \leq ||G||$. Then

$$
D''(FG) = D''\left(w^* - \lim_\alpha w^* - \lim_\beta a_\alpha b_\beta\right)
$$

$$
= w^* - \lim_\alpha w^* - \lim_\beta D''(a_\alpha b_\beta)
$$

$$
= w^* - \lim_\alpha w^* - \lim_\beta \left(\tau(a_\alpha)D(b_\beta) + D(a_\alpha)\sigma(b_\beta)\right)
$$

$$
= \tau''(F)D''(G) + D''(F)\sigma''(G)
$$

and so $D''$ is a $(\sigma'', \tau'')$-derivation.

Now we are ready to state some equivalent conditions by $(\sigma, \tau)$-amenability of Banach algebras.
Theorem 2.1. Let $\sigma$ and $\tau$ be two continuous homomorphisms on Banach algebra $\mathcal{A}$. The following statements are equivalent:

1. $\mathcal{A}$ is ($\sigma, \tau$)-amenable.

2. For each Banach algebra $\mathcal{B}$ and every homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, $H_{(\sigma, \tau)}^{1}(\mathcal{A}, \mathcal{B}^{\varphi}) = 0$.

3. For each Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, $H_{(\sigma, \tau)}^{1}(\mathcal{A}, \mathcal{B}^{\varphi}) = 0$.

4. For each Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, if $d : \mathcal{A} \rightarrow \mathcal{B}^{\varphi}$ is a ($\sigma, \tau$)-derivation satisfies
   \[ (d(a))(\varphi(b)) + (d(b))(\varphi(a)) = 0 \quad (a, b \in \mathcal{A}), \]
   then $d$ is ($\sigma, \tau$)-inner derivation.

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). It is sufficient to show that (4) $\Rightarrow$ (1). Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \rightarrow \mathcal{X}^{*}$ be a ($\sigma, \tau$)-derivation. Set $\mathcal{B} = \mathcal{A} \oplus_{1} \mathcal{X}$ and define injective homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ by $\varphi(a) = (a, 0)$ and so we can assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. Define $d : \mathcal{A} \rightarrow \mathcal{B}^{\varphi}$ by $d(a) = (0, D(a))$. The map $d$ is ($\sigma, \tau$)-derivation, since

\[
\begin{align*}
    d(ab) &= (0, D(ab)) = (0, D(a)\sigma(b) + \tau(a)D(b)) \\
    &= (0, D(a))(0, \sigma(b)) + (0, \tau(a))(0, D(b)) \\
    &= d(a)\varphi(\sigma(b)) + \varphi(\tau(a))d(b) \\
    &= d(a)\cdot \sigma(b) + \tau(a)\cdot d(b) \quad (a, b \in \mathcal{A}).
\end{align*}
\]

Since $d(a)\varphi(\sigma(b)) + d(b)\varphi(\tau(a)) = (0, D(a))(0, \sigma(b)) + (0, D(b))(0, \tau(a)) = 0$, we have $d(a)\varphi(\sigma(b)) + d(b)\varphi(\tau(a)) = 0$. Therefore, $d$ is ($\sigma, \tau$)-inner derivation.
It follows from our assumption that \( d \) is a \((\sigma, \tau)\)-inner derivation. Hence there are \( f \in \mathcal{A}^* \) and \( g \in \mathcal{X}^* \) such that

\[
(0, D(a)) = d(a) = (\sigma(a), 0)(f, g) - (f, g)(\tau(a), 0) = (\sigma(a)f - f\tau(a), \sigma(a)g - g\tau(a)).
\]

Thus \( D(a) = \sigma(a)g - g\tau(a) \), hence \( D \) is \((\sigma, \tau)\)-inner derivation.

**Definition 2.1.** Let \( \mathcal{A} \) be a Banach algebra and \( \sigma \) be a continuous homomorphism on \( \mathcal{A} \). The Banach algebra \( \mathcal{A} \) is called approximately \( \sigma \)-contractible, if for each Banach \( \mathcal{A} \)-bimodule \( \mathcal{X} \) and \( \sigma \)-derivation \( D : \mathcal{A} \rightarrow \mathcal{X} \), there exists a bounded net \( (x_a) \subseteq \mathcal{X} \) such that

\[
D(a) = \lim_{\alpha} \left( \sigma(a)x_a - x_a\sigma(a) \right) \quad (a \in \mathcal{A}).
\]

In the following theorem we follow the structure of Proposition 2.8.59 [1].

**Theorem 2.2.** Let \( \mathcal{A} \) be a Banach algebra and \( \sigma \) be a bounded homomorphism on \( \mathcal{A} \). Then the following assertions are equivalent:

1. \( \mathcal{A} \) is \( \sigma \)-amenable.

2. For every \( \mathcal{A} \)-bimodule \( \mathcal{X} \), \( H^1_{(\sigma, \sigma)}(\mathcal{A}, \mathcal{X}^{**}) = 0 \)

3. \( \mathcal{A} \) is approximately \( \sigma \)-contractible.

**Proof.** (1) \( \Rightarrow \) (2) is trivially. (2) \( \Rightarrow \) (3): Let \( D : \mathcal{A} \rightarrow \mathcal{X} \) be a \( \sigma \)-derivation from \( \mathcal{A} \) into \( \mathcal{A} \)-bimodule \( \mathcal{X} \) and let \( J_\mathcal{X} : \mathcal{X} \rightarrow \mathcal{X}^{**} \) be the canonical embedding, then for each \( a, b \in \mathcal{A} \) we have

\[
\tilde{D}(ab) = (J_\mathcal{X} \circ D)(ab) = J_\mathcal{X} \left( \sigma(a)D(b) + D(a)\sigma(b) \right)
\]
where $\tilde{D}$ is a $\sigma$-derivation. Then by (2) there exists $\Lambda \in \mathcal{X}^{**}$ such that $\tilde{D}(a) = \sigma(a)\Lambda - \Lambda \sigma(a)$ $(a \in \mathcal{A})$. Set $m = ||\Lambda||, \mathcal{U} = \mathcal{X}_{[m]}$. Then $\Lambda \in J_{\mathcal{X}}(\mathcal{U})^{w^*}$. Let $a_1, a_2, a_3, \ldots, a_n \in \mathcal{A}$, then $\mathcal{V} = \cap_{j=1}^{n}\left(\sigma(a_j)\mathcal{U} - \mathcal{U}\sigma(a_j)\right)$ is a convex subset of $\mathcal{X}^{(n)}$ and $(D(a_1), D(a_2), \ldots, D(a_n)) \in \mathcal{V}^{weak^*}$. Thus for each finite subset $F$ of $\mathcal{A}$, and $\varepsilon > 0$, there exists $x_{(F, \varepsilon)} \in \mathcal{U}$ such that

\[
||D(a) - (\sigma(a)x_{(F, \varepsilon)} - x_{(F, \varepsilon)}\sigma(a))|| < \varepsilon \quad (a \in F).
\]

The family of such pairs $(F, \varepsilon)$ is a directed if order $\leq$ is given by

\[
(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \iff F_1 \subseteq F_2, \varepsilon_1 \leq \varepsilon_2.
\]

Also we have

\[
D(a) = \lim_{(F, \varepsilon)} \left(\sigma(a)x_{(F, \varepsilon)} - x_{(F, \varepsilon)}\sigma(a)\right).
\]

(3) $\implies$ (1): Let $D : \mathcal{A} \to \mathcal{X}^*$ be a $\sigma$-derivation. Then there exists a net $(x_{a}^{\prime}) \subseteq \mathcal{X}^*$ such that $D(a) = \lim_{a} \left(\sigma(a)x_{a}^\prime - x_{a}^\prime\sigma(a)\right)$ $(a \in \mathcal{A})$. By passing to a subnet we may assume that $w^* - \lim x_{a}^\prime = x^\prime$ in $\mathcal{X}^*$ and then $D(a) = \sigma(a)x^\prime - x^\prime\sigma(a)$. Thus $\mathcal{A}$ is $\sigma$-amenable.

**Theorem 2.3.** Let $\mathcal{A}$ be a Banach algebra and $\sigma$ be a continuous homomorphism on $\mathcal{A}$. If $\mathcal{A}^{**}$ is $\sigma''$-amenable, then $\mathcal{A}$ is $\sigma$-amenable.

**Proof.** Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $D : \mathcal{A} \to \mathcal{X}^{**}$ be a $\sigma$-derivation. Then by Lemma 2.1, $D'' : \mathcal{A}^{**} \to \mathcal{X}^{****}$ is a $\sigma''$-derivation. Since $\mathcal{A}^{**}$ is $\sigma''$-amenable, then there exists $x^{(4)} \in \mathcal{X}^{****}$ such that $D''(a'') = \sigma''(a'')x^{(4)} - x^{(4)}\sigma''(a'')$, $(a'' \in \mathcal{A}^{**})$. We have $\mathcal{X}^{****} = \mathcal{X}^{**} \oplus (\mathcal{X}^*)^\perp$ (as $\mathcal{A}^{**}$-bimodules). Let $P : \mathcal{X}^{****} \to \mathcal{X}^{**}$ be the natural projection. Then for each $a \in \mathcal{A}$, we have $D(a) = \sigma(a)P(x^{(4)}) - P(x^{(4)})\sigma(a)$, and so $D \in N_{(\sigma, \sigma)}(\mathcal{A}, \mathcal{X}^{**})$. Thus by above theorem, $\mathcal{A}$ is $\sigma$-amenable.
In the following we fined an easy equivalent condition for $\sigma$-amenability of a Banach algebra.

**Proposition 2.1.** Let $\mathcal{A}$ be a Banach algebra and let $\sigma$ be a continuous homomorphism on $\mathcal{A}$. Then $\mathcal{A}$ is a $\sigma$-amenable if and only if for every Banach algebra $\mathcal{B}$ and every injective homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$, $H^1_{(\sigma, \sigma)}(\mathcal{A}, \mathcal{B}^{**}) = 0$.

**Proof.** One side is clear, so we prove the other side. Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to \mathcal{X}^{**}$ be a $\sigma$-derivation. If $\phi : \mathcal{A} \to \mathcal{A} \oplus_1 \mathcal{X}$ is defined by $\varphi(a) = (a, 0)$. Then $\varphi$ is injective and $\varphi^{**} : \mathcal{A}^{**} \to (\mathcal{A} \oplus_1 \mathcal{X})^{**}$ the second transpose of $\varphi$ is a Banach algebra homomorphism and $((\mathcal{A} \oplus_1 \mathcal{X})_{\varphi})^{**} \cong (\mathcal{A}^{**} \oplus_1 \mathcal{X}^{**})_{\varphi^{**}}$ as $\mathcal{A}^{**}$-bimodules. Then

$$H^1_{(\sigma, \sigma)}(\mathcal{A}, (\mathcal{A}^{**} \oplus_1 \mathcal{X}^{**})_{\varphi^{**}}) = H^1_{(\sigma, \sigma)}((\mathcal{A} \oplus_1 \mathcal{X})_{\varphi})^{**} = \{0\}.$$  \hspace{1cm} (2.2)

Now we define $D_1 : \mathcal{A} \to \mathcal{A}^{**} \oplus_1 \mathcal{X}^{**}$ by $D_1(a) = (0, D(a))$. For $a, b \in \mathcal{A}$ we have $D_1(ab) = D_1(a)\varphi^{**}(b) + \varphi^{**}(a)D_1(b)$. Thus $D_1$ is a $\sigma$-derivation from $\mathcal{A}$ into $(\mathcal{A}^{**} \oplus_1 \mathcal{X}^{**})_{\varphi^{**}}$. By (2.2), $D_1$ is $\sigma$-inner. Therefore there exist $a'' \in \mathcal{A}^{**}, x'' \in \mathcal{X}^{**}$ such that

$$(0, D(a)) = D_1(a) = (a'', x'')(0, \sigma(a)) - (0, \sigma(a))(a'', x''),$$

Thus $D$ is $\sigma$-inner. Therefore $H^1_{(\sigma, \sigma)}(\mathcal{A}, \mathcal{X}^{**}) = 0$, and by Theorem 2.2, $\mathcal{A}$ is $\sigma$-amenable.

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**References**


