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Abstract. In this paper, we introduce \( \phi \)-prime and \( \phi \)-primary elements in an \( L \)-module \( M \). Many of its characterizations and properties are obtained. By counter examples, it is shown that a \( \phi \)-prime element of \( M \) need not be prime, a \( \phi \)-primary element of \( M \) need not be \( \phi \)-prime, a \( \phi \)-primary element of \( M \) need not be prime and a \( \phi \)-primary element of \( M \) need not be primary. Finally, some results for almost prime and almost primary elements of an \( L \)-module \( M \) with their characterizations are obtained. Also, we introduce the notions of \( n \)-potent prime(respectively \( n \)-potent primary) elements in \( L \) and \( M \) to obtain interrelations among them where \( n \geq 2 \).

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1. Introduction

In multiplicative lattices, the study of \( \phi \)-prime and \( \phi \)-primary elements is done by C. S. Manjarekar and A. V. Bingi in [16]. Our aim is to extend the notion of \( \phi \)-prime and \( \phi \)-primary elements in a multiplicative lattice to the notion of \( \phi \)-prime and \( \phi \)-primary elements in a lattice module and study its properties. According to [1], a proper element \( N \) of an \( L \)-module \( M \) is said to be prime if for all \( A \in M, a \in L, aA \leq N \) implies either \( A \leq N \) or \( a \leq (N : I_M) \). According to [10], a proper element \( N \) of an \( L \)-module \( M \) is said to be primary if for all \( A \in M, a \in L, aA \leq N \) implies either \( A \leq N \) or \( a \leq \sqrt{N : I_M} \). By restricting where \( aA \) lies, weakly prime and weakly primary elements in lattice modules are studied by C. S. Manjarekar et. al. in [19] and [20], respectively. A proper element \( N \) of an \( L \)-module \( M \) is said to be weakly prime if for all \( A \in M, a \in L, O_M \neq aA \leq N \) implies either \( A \leq N \) or \( a \leq (N : I_M) \). A proper element \( N \) of an \( L \)-module \( M \) is said to be weakly primary if for all \( A \in M, a \in L, O_M \neq aA \leq N \) implies either \( A \leq N \) or \( a \leq (N : I_M) \).
or \( a \leq \sqrt{N} : I_M \). Keeping this in mind, in this paper we define and study \( \phi \)-prime and \( \phi \)-primary elements of an \( L \)-module \( M \).

A multiplicative lattice \( L \) is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element \( e \in L \) is called meet principal if \( a \land be = ((a : e) \land b)e \) for all \( a,b \in L \). An element \( e \in L \) is called join principal if \( (ae \lor b) : e = (b : e) \lor a \) for all \( a,b \in L \). An element \( e \in L \) is called principal if \( e \) is both meet principal and join principal. An element \( a \in L \) is called compact if for \( X \subseteq L \), \( a \leq \lor X \) implies the existence of a finite number of elements \( a_1, a_2, \cdots, a_n \) in \( X \) such that \( a \leq a_1 \lor a_2 \lor \cdots \lor a_n \). The set of compact elements of \( L \) will be denoted by \( L_c \). If each element of \( L \) is a join of compact elements of \( L \), then \( L \) is called a compactly generated lattice or simply a CG-lattice. \( L \) is said to be a principally generated lattice or simply a PG-lattice if each element of \( L \) is a join of principal elements of \( L \).

Throughout this paper, \( L \) denotes a compactly generated multiplicative lattice with greatest compact element 1 in which every finite product of compact elements is compact.

An element \( a \in L \) is said to be proper if \( a < 1 \). A proper element \( m \in L \) is said to be maximal if for every element \( x \in L \) such that \( m < x \leq 1 \) implies \( x = 1 \). A proper element \( p \in L \) is called a prime element if \( ab \leq p \) implies \( a \leq p \) or \( b \leq p \) where \( a,b \in L \) and is called a primary element if \( ab \leq p \) implies \( a \leq p \) or \( b^n \leq p \) for some \( n \in \mathbb{Z}_+ \) where \( a,b \in L_* \). For \( a,b \in L \), \( (a : b) = \lor \{ x \in L \mid xb \leq a \} \). The radical of \( a \in L \) is denoted by \( \sqrt{a} \) and is defined as \( \lor \{ x \in L_* \mid x^n \leq a \}, \) for some \( n \in \mathbb{Z}_+ \). A multiplicative lattice is called a Noether lattice if it is modular, principally generated and satisfies the ascending chain condition.

A proper element \( a \in L \) is said to be nilpotent if \( a^n = 0 \) for some \( n \in \mathbb{Z}_+ \). According to [9], a proper element \( p \in L \) is said to be almost prime if for all \( a,b \in L \), \( ab \leq p \) and \( ab \not\leq p^2 \) implies either \( a \leq p \) or \( b \leq p \) and according to [15], a proper element \( p \in L \) is said to be almost primary if for all \( a,b \in L \), \( ab \leq p \) and \( ab \not\leq p^2 \) implies either \( a \leq p \) or \( b \leq \sqrt{p} \). Further study on almost prime and almost primary elements of a multiplicative lattice \( L \) is seen in [16], [5] and [4]. According to [12], a proper element \( q \in L \) is said to be 2-absorbing if for all \( a,b,c \in L \), \( abc \leq q \) implies either \( ab \leq q \) or \( bc \leq q \) or \( ca \leq q \). According to [18], a proper element \( q \in L \) is said to be 2-absorbing primary if for all \( a,b,c \in L \), \( abc \leq q \) implies either \( ab \leq q \) or \( bc \leq \sqrt{q} \) or \( ca \leq \sqrt{q} \). The reader is referred to [2], [3] and [9] for general background and terminology in multiplicative lattices.

Let \( M \) be a complete lattice and \( L \) be a multiplicative lattice. Then \( M \) is called \( L \)-module or module over \( L \) if there is a multiplication between elements of \( L \) and \( M \) written as \( ab \) where \( a \in L \) and \( b \in M \) which satisfies the following properties:

1. \((\forall a_\alpha)A = \bigvee a_\alpha A \),
2. \( a(\forall A_\alpha) = \bigvee (a A_\alpha) \),
3. \((ab)A = a(bA) \),
4. \( 1A = A \),
5. \( 0A = O_M \), for all \( a,a_\alpha \), \( b \in L \) and \( A,A_\alpha \in M \) where 1 is the supremum of \( L \) and 0 is the infimum of \( L \). We denote by \( O_M \) and \( I_M \) for the least element and the greatest element of \( M \), respectively. Elements of \( L \) will generally be denoted by \( a,b,c, \cdots \) and elements of \( M \) will generally be denoted by \( A,B,C, \cdots \).

Let \( M \) be an \( L \)-module. For \( N \in M \) and \( a \in L \), \( (N : a) = \bigvee \{ X \in M \mid aX \leq N \} \). For \( A,B \in M \), \( (A : B) = \bigvee \{ x \in L \mid xB \leq A \} \). If \( (O_M : I_M) = 0 \), then \( M \) is called a faithful \( L \)-module. \( M \) is called a torsion free \( L \)-module if for all \( c \in L \), \( B \in M \), \( cB = O_M \) implies either \( B = O_M \) or \( c = 0 \). An \( L \)-module \( M \) is called a multiplication lattice module if for
every element \( N \in M \) there exists an element \( a \in L \) such that \( N = aI_M \). By proposition 3 in [10], an \( L \)-module \( M \) is a multiplication lattice module if and only if \( N = (N : I_M)I_M \) \( \forall N \in M \). An element \( N \in M \) is called meet principal if \( (b \wedge (B : N))N = bN \wedge B \) for all \( b \in L, B \in M \). An element \( N \in M \) is called join principal if \( b \vee (B : N) = (bN \vee B) : N \) for all \( b \in L, B \in M \). An element \( N \in M \) is said to be principal if \( N \) is both meet principal and join principal. \( M \) is said to be a PG-lattice \( L \)-module if each element of \( M \) is a join of principal elements of \( M \). An element \( N \in M \) is called compact if \( N \leq \bigvee_{\alpha} A_{\alpha} \) implies \( N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n} \) for some finite subset \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \). The set of compact elements of \( M \) is denoted by \( M_c \). If each element of \( M \) is a join of compact elements of \( M \), then \( M \) is called a CG-lattice \( L \)-module. An element \( N \in M \) is said to be proper if \( N < I_M \). A proper element \( N \in M \) is said to be maximal if whenever there exists an element \( B \in M \) such that \( N \leq B \) then either \( N = B \) or \( B = I_M \). If a proper element \( N \in M \) is prime, then \( (N : I_M) \in L \) is prime. A proper element \( N \in M \) is said to be a radical element if \( (N : I_M) = \sqrt{N : I_M} \). An \( L \)-module \( M \) is said to be Noetherian, if \( M \) satisfies the ascending chain condition, is modular and is principally generated. According to [17], a proper element \( Q \) of an \( L \)-module \( M \) is said to be 2-absorbing if for all \( a, b \in L, N \in M \), \( abN \leq Q \) implies either \( aN \leq \sqrt{Q : I_M} \) or \( bN \leq Q \) or \( aN \leq Q \). According to [6], a proper element \( Q \) of an \( L \)-module \( M \) is said to be 2-absorbing primary if for all \( a, b \in L, N \in M \), \( abN \leq Q \) implies either \( aN \leq \sqrt{Q : I_M}I_M \) or \( bN \leq \sqrt{Q : I_M}I_M \) or \( aN \leq \sqrt{Q : I_M}I_M \). The reader is referred to [1], [10] and [14] for terminology in lattice modules.

This paper is motivated by [24] and [7]. Many of the results obtained in this paper are lattice module version of the results in [16] and principal elements of \( M \) are used wherever needed with some more conditions on \( M \). First section of this paper is comprised of \( \phi \)-prime and \( \phi \)-primary elements of an \( L \)-module \( M \). Second section is comprised of almost prime and almost primary elements of an \( L \)-module \( M \). By counter examples, it is shown that a \( \phi \)-prime element of \( M \) need not be prime (see Example 1), a \( \phi \)-primary element of \( M \) need not be \( \phi \)-prime (see Example 2), a \( \phi \)-primary element of \( M \) need not be prime (see Example 3) and a \( \phi \)-primary element of \( M \) need not be primary (see Example 4). We define 2-potent prime and 2-potent primary elements in an \( L \)-module \( M \). By counter examples, it is shown that an almost primary element of \( M \) need not be 2-potent prime (see Example 5) and a 2-potent prime element of \( M \) which is almost primary need not be prime (see Example 6). Also, we introduce the notions of \( n \)-potent prime and \( n \)-potent primary elements in an \( L \)-module \( M \) where \( n \geq 2 \). We find condition(s) under which a \( \phi \)-prime element of \( M \) is prime (see Theorems 5-10). Also, we find condition(s) under which a \( \phi \)-primary element of \( M \) is prime (see Theorems 15-23). Absorbing concepts in an \( L \)-module \( M \) are related to these notions of \( \phi \)-prime and \( \phi \)-primary in \( M \). In the last section of this paper, many characterizations of almost prime and almost primary elements of \( M \) are obtained. By a counter example, it is shown that an almost primary element of \( M \) need not be idempotent (see Example 7). By a counter example, it is shown that an almost primary element of \( M \) need not be weakly primary (see Example 8). Finally, we show that if an element in \( M \) is almost prime (respectively almost primary), then its corresponding element in \( L \) is also almost prime (respectively almost primary) and vice
versa.

2. $\phi$-Prime and $\phi$-Primary Elements in $M$

The study of weakly prime and weakly primary elements of an $L$-module $M$ is carried out by A. V. Bingi and C. S. Manjarekar in [8]. Also, the notion of an almost prime element of an $L$-module $M$ is seen in [22]. With weakly prime elements and almost prime elements of an $L$-module $M$ in mind, we begin with introducing the notion of a $\phi$-prime element of an $L$-module $M$.

**Definition 1.** Let $\phi : M \rightarrow M$ be a function on an $L$-module $M$. A proper element $N \in M$ is said to be $\phi$-prime if for all $a \in L$, $A \in M$, $aA \leq N$ and $aA \nleq \phi(N)$ implies either $A \leq N$ or $a \nleq (N : IM)$.

Now if $\phi_\alpha : M \rightarrow M$ is a function on an $L$-module $M$, then $\phi_\alpha$-prime elements of $M$ are defined by following settings in the Definition 1 of a $\phi$-prime element.

- $\phi_0(N) = O_M$. Then $N \in M$ is called a weakly prime element.
- $\phi_2(N) = (N : IM)N$. Then $N \in M$ is called a 2-almost prime element or a $\phi_2$-prime element or simply an almost prime element.
- $\phi_n(N) = (N : IM)^{n-1}N$ ($n \geq 2$). Then $N \in M$ is called an $n$-almost prime element or a $\phi_n$-prime element ($n \geq 2$).
- $\phi_\omega(N) = \bigwedge_{i=1}^\infty (N : IM)^iN$. Then $N \in M$ is called an $\omega$-prime element or $\phi_\omega$-prime element.

Since $N \setminus \phi(N) = N \setminus (N \wedge \phi(N))$, so without loss of generality, throughout this paper, we assume that $\phi(N) \leq N$.

**Definition 2.** Given two functions $\gamma_1, \gamma_2 : M \rightarrow M$ on an $L$-module $M$, we define $\gamma_1 \leq \gamma_2$ if $\gamma_1(N) \leq \gamma_2(N)$ for all $N \in M$.

Clearly, we have the following order:

$$\phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$$

Now before obtaining the characterizations of a $\phi$-prime element of an $L$-module $M$, we state the following essential lemma which is outcome of Lemma 2.3.13 from [11].

**Lemma 1.** Let $a_1, a_2 \in L$. Suppose $b \in L$ satisfies the following property:

(*) If $h \in L^+$ with $h \leq b$, then either $h \leq a_1$ or $h \leq a_2$.

Then either $b \leq a_1$ or $b \leq a_2$.

**Theorem 1.** Let $M$ be a CG-lattice $L$-module, $N \in M$ be a proper element and $\phi : M \rightarrow M$ be a function on $M$. Then the following statements are equivalent:

1. $N$ is a $\phi$-prime element of $M$. 

2. For every $A \in M$ such that $A \notin N$, either $(N : A) = (N : I_M)$ or $(N : A) = (\phi(N) : A)$.

3. For every $r \in L$ such that $r \notin (N : I_M)$, either $(N : r) = N$ or $(N : r) = (\phi(N) : r)$.

4. For every $r \in L_*$, $A \in M_*$, if $rA \leq N$ and $rA \notin \phi(N)$, then either $r \leq (N : I_M)$ or $A \leq N$.

Proof. ①⇒②. Suppose ① holds. Let $A \in M$ be such that $A \notin N$. Obviously, $(\phi(N) : A) \leq (N : A)$ and $(N : I_M) \leq (N : A)$. Let $a \in L_*$ be such that $a \leq (N : A)$. Then $aA \leq N$. If $aA \leq \phi(N)$, then $a \leq (\phi(N) : A)$. If $aA \notin \phi(N)$, then since $N$ is $\phi$-prime and $A \notin N$, it follows that $a \leq (N : I_M)$. Hence by Lemma 1, either $(N : A) \leq (\phi(N) : A)$ or $(N : A) \leq (N : I_M)$. Thus either $(N : A) = (\phi(N) : A)$ or $(N : A) = (N : I_M)$.

②⇒③. Suppose ② holds. Let $r \notin (N : I_M)$ for $r \in L$. Then $rI_M \notin N$. Using ②, we have either $(N : rI_M) = (N : I_M)$ or $(N : rI_M) = (\phi(N) : rI_M)$. Now let $K \leq (N : r)$ for $K \in M_*$. As $(K : I_M)I_M \leq K$, we have, $(K : I_M)I_M \leq (N : r)$ and $(K : I_M)I_M \in M_*$. Clearly, $K \leq (N : r)$ implies $(K : I_M) \leq (N : r) : I_M = (N : rI_M)$. Thus either $(K : I_M)I_M \leq N$ or $(K : I_M)I_M \leq (\phi(N) : r)$, which implies that either $(N : r) \leq N$ or $(N : r) \leq (\phi(N) : r)$, by Lemma 3.1 of [22]. Since $rN \leq N$ gives $N \leq (N : r)$ and $\phi(N) \leq N$ gives $(\phi(N) : r) \leq (N : r)$, it follows that either $(N : r) = N$ or $(N : r) = (\phi(N) : r)$.

③⇒④. Suppose ③ holds. Let $rA \leq N$, $rA \notin \phi(N)$ and $r \notin (N : I_M)$ for $r \in L_*$, $A \in M_*$. Then by ③, we have either $(N : r) = (\phi(N) : r)$ or $(N : r) = N$. If $(N : r) = (\phi(N) : r)$, then as $rA \leq N$, it follows that $A \leq (\phi(N) : r)$ which contradicts $rA \notin \phi(N)$ and so we must have $(N : r) = N$. Therefore $rA \leq N$ gives $A \leq N$.

④⇒①. Suppose ④ holds. Let $aQ \leq N$, $aQ \notin \phi(N)$ and $Q \notin N$ for $a \in L$, $Q \in M$. As $L$ and $M$ are compactly generated, there exist $x' \in L_*$ and $Y$, $Y' \in M_*$ such that $x' \leq a$, $Y \leq Q$, $Y' \leq Q$, $Y' \notin N$ and $x'Y' \notin \phi(N)$. Let $x \in L_*$ be such that $x \leq a$. Then $(x \vee x') \leq (x \vee Y') \leq (x \vee x')(Y \vee Y') \leq aQ \leq N$, $(x \vee x')(Y \vee Y') \notin \phi(N)$ and $(Y \vee Y') \notin N$. So by ④, $(x \vee x') \leq (N : I_M)$ which implies $a \leq (N : I_M)$. Therefore $N$ is $\phi$-prime.

The following 2 corollaries are consequences of Theorem 1.

Corollary 1. Let $M$ be a CG-lattice $L$-module and $N \in M$ be a proper element. Then the following statements are equivalent:

① $N$ is a weakly prime element of $M$.

② For every $A \in M$ such that $A \notin N$, either $(N : A) = (N : I_M)$ or $(N : A) = (O_M : A)$.

③ For every $r \in L$ such that $r \notin (N : I_M)$, either $(N : r) = N$ or $(N : r) = (O_M : r)$.

④ For every $r \in L_*$, $A \in M_*$, if $O_M \neq rA \leq N$, then either $r \leq (N : I_M)$ or $A \leq N$. 
Corollary 2. Let $M$ be a CG-lattice $L$-module and $N \in M$ be a proper element. Then the following statements are equivalent:

1. $N$ is an almost prime element of $M$.
2. For every $A \in M$ such that $A \not\leq N$, either $(N : A) = ((N : I_M) N : A)$ or $(N : A) = (N : I_M)$.
3. For every $r \in L$ such that $r \not\leq (N : I_M)$, either $(N : r) = ((N : I_M) N : r)$ or $(N : r) = N$.
4. For every $r \in L$, $A \in M$, if $rA \not\leq N$ and $rA \not\leq (N : I_M) N$, then either $A \leq N$ or $r \leq (N : I_M)$.

To obtain the relation among prime, weakly prime, $\omega$-prime, $n$-almost prime ($n \geq 2$) and almost prime elements of an $L$-module $M$, we prove the following result.

Theorem 2. Let $\gamma_1, \gamma_2 : M \longrightarrow M$ be functions on an $L$-module $M$ such that $\gamma_1 \leq \gamma_2$. Then every proper $\gamma_1$-prime element of $M$ is $\gamma_2$-prime.

Proof. Let a proper element $N \in M$ be $\gamma_1$-prime. Assume that $aA \leq N$ and $aA \not\leq \gamma_2(N)$ for $a \in L$, $A \in M$. Then as $\gamma_1 \leq \gamma_2$, we have $aA \not\leq \gamma_1(N)$. Since $N$ is $\gamma_1$-prime, it follows that either $A \leq N$ or $a \leq (N : I_M)$ and hence $N$ is $\gamma_2$-prime.

Theorem 3. Let $N$ be a proper element of an $L$-module $M$. Then $N$ is prime implies $N$ is weakly prime, $N$ is weakly prime implies $N$ is $\omega$-prime, $N$ is $\omega$-prime implies $N$ is $n$-almost prime ($n \geq 2$) and $N$ is $n$-almost prime ($n \geq 2$) implies $N$ is almost prime.

Proof. By definition, every prime element of an $L$-module $M$ is weakly prime and hence $N$ is prime implies $N$ is weakly prime. The remaining implications follow by using Theorem 2 to the fact that $\phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2$.

From the Theorem 3, we get the following characterization of a $\omega$-prime element of an $L$-module $M$.

Corollary 3. Let $N$ be a proper element of an $L$-module $M$. Then $N$ is $\omega$-prime if and only if $N$ is $n$-almost prime for every $n \geq 2$.

Proof. Assume that $N \in M$ is $n$-almost prime for every $n \geq 2$. Let $aA \leq N$ and $aA \not\leq \bigwedge_{i=1}^{\infty} (N : I_M)^iN$ for $a \in L$, $A \in M$. Then $aA \not\leq (N : I_M)^{n-1}N$ for some $n \geq 2$. Since $N$ is $n$-almost prime, we have either $a \leq (N : I_M)$ or $A \leq N$ and hence $N$ is $\omega$-prime. The converse follows from Theorem 3.

Before going to the characterization of an $n$-almost prime element of an $L$-module $M$, we recall the definition of the Jacobson radical of $L$. According to [2], in a multiplicative lattice $L$ with 1 compact, the Jacobson radical is the element $\bigwedge \{m \in L \mid m$ is a maximal element $\}$. 
Theorem 4. Let $L$ be a Noether lattice, $M$ be a torsion free Noetherian $L$-module and $f \in L$ be the Jacobson radical. Then a proper element $N \in M$ such that $(N : I_M) \leq f$ is $n$-almost prime for every $n \geq 2$ if and only if $N$ is prime.

Proof. Assume that $N \in M$ is $n$-almost prime where $n \geq 2$. Let $aA \leq N$ for $a \in L$, $A \in M$. If $aA \not\leq (N : I_M)^{n-1}N$ for $n \geq 2$, then as $N$ is $n$-almost prime, we have either $A \leq N$ or $a \leq (N : I_M)$. If $aA \leq (N : I_M)^{n-1}N$ for all $n \geq 2$, then as $(N : I_M) \leq f$, from Corollary 3.3 of [13], it follows that $aA \leq \bigwedge_{n=1}^{\infty} (N : I_M)^n N = O_M$ and thus $aA = O_M$. Since $M$ is torsion free, we have either $A = O_M$ or $a = 0$ which implies either $A \leq N$ or $a \leq (N : I_M)$ and hence $N$ is prime. The converse follows from Theorem 3.

Clearly, every prime element of an $L$-module $M$ is $\phi$-prime. But the converse is not true which is shown in the following example by taking $\phi(N) = (N : I_M)N$ for convenience.

Example 1. If $Z$ is the ring of integers, then $Z_{24}$ is a $Z$-module. Assume that $(k)$ denotes the cyclic ideal of $Z$ generated by $k \in Z$ and $<\mathbf{t}>$ denotes the cyclic submodule of $Z$-module $Z_{24}$ where $\mathbf{t} \in Z_{24}$. Suppose that $L = L(Z)$ is the set of all ideals of $Z$ and $M = L(Z_{24})$ is the set of all submodules of $Z$-module $Z_{24}$. The multiplication between elements of $L$ and $M$ is given by $(k_i) <\mathbf{t}_j> = <\mathbf{k}_i\mathbf{t}_j>$ for every $(k_i) \in L$ and $<\mathbf{t}_j>\in M$ where $k_i, t_j \in Z$. Then $M$ is a lattice module over $L$ [22], Example 2.5]. Let $N$ be the cyclic submodule of $M$ generated by $5$. It is easy to see that $O_M = <\mathbf{0}> = N$ is weakly prime and hence almost prime ($\phi_2$-prime) while $N$ is not prime, since $(2) <\mathbf{t}_2> \leq N$ but $<\mathbf{t}_2> \not\leq N$ and $(2) \not\leq (N : I_M) = (0)$ where $I_M = <\mathbf{t}>$.

Now we obtain six results that show under which condition(s) a $\phi$-prime element of an $L$-module $M$ is prime. But before that we prove the required cancellation laws of $M$ in the form of following lemmas.

Lemma 2. Let $M$ be a torsion free $L$-module and $O_M \neq A \in M$ be a weak join principal element. Then $aA \leq bA$ implies $a \leq b$ for $a, b \in L$ where $b \neq 0$.

Proof. Let $aA \leq bA$ and $O_M \neq A \in M$ be a weak join principal element for $a, b \in L$. As $M$ is a torsion free $L$-module, we have $(O_M : A) = 0$. Then clearly, $a = a \lor 0 = a \lor (O_M : A) = (aA : A) \leq (bA : A) = b \lor (O_M : A) = b \lor 0 = b$ which implies $a \leq b$.

Lemma 3. Let $M$ be a torsion free $L$-module and $O_M \neq A \in M$ be a weak join principal element. Then $aA = bA$ implies $a = b$ for $a, b \in L$ where $a \neq 0, b \neq 0$.

Proof. The proof is obvious.

Now we have a characterization of a $\phi$-prime element of an $L$-module $M$.

Theorem 5. Let $M$ be a torsion free $L$-module and $O_M \neq N < I_M$ be a weak join principal element of $M$. Then $N$ is $\phi$-prime for some $\phi \leq \phi_2$ if and only if $N$ is prime.
Proof. Assume that \( N \in M \) is a prime element. Then obviously, \( N \) is \( \phi \)-prime for every \( \phi \) and hence for some \( \phi \leq \phi_2 \). Conversely, let \( N \) be \( \phi \)-prime for some \( \phi \leq \phi_2 \). Then by Theorem 2, \( N \) is \( \phi_2 \)-prime. Let \( aA \leq N \) for \( a \in L, \ A \in M \). If \( aA \notin \phi_2(N) \), then as \( N \) is \( \phi_2 \)-prime, we have either \( A \not\leq N \) or \( a \leq (N : I_M) \). Next, assume that \( aA \leq \phi_2(N) \). If \( a(A \vee N) \notin \phi_2(N) \), then as \( a(A \vee N) \leq N \) and \( N \) is \( \phi_2 \)-prime, we have either \( (A \vee N) \leq N \) or \( a \leq (N : I_M) \) and hence either \( A \not\leq N \) or \( a \leq (N : I_M) \). Finally, if \( a(A \vee N) \leq \phi_2(N) \), then \( aN \leq (N : I_M)N \) which implies \( a \leq (N : I_M) \), by Lemma 2 and hence \( N \) is prime.

Now we show that the Theorem 5 can also be achieved by changing the conditions on \( M \) and \( L \). According to [23], in a Noether lattice \( L \), an element \( a \in L \) is said to satisfy the restricted cancellation law (RCL) if for all \( b, c \), \( \phi \)-prime, we have either \( A \not\leq N \) or \( a \leq (N : I_M) \). According to [23], in a Noether lattice \( L \), an element \( a \in L \) is said to satisfy the restricted cancellation law (RCL) if for all \( b, c \), \( a \in L \), \( a \) is \( \phi \)-prime for some \( \phi \leq \phi_2 \) and hence for some \( \phi \leq \phi_2 \).

Theorem 6. Let \( L \) be a Noether PG-lattice and \( M \) be a faithful multiplication PG-lattice \( L \)-module with \( I_M \) compact. Let \( N \) be a proper element of \( M \) such that \( 0 \neq (N : I_M) \leq L \) satisfies the restricted cancellation law (RCL) and is a non-nilpotent element. Then \( N \) is \( \phi \)-prime for some \( \phi \leq \phi_2 \) if and only if \( N \) is prime.

Proof. Assume that \( N \in M \) is a prime element. Then obviously, \( N \) is \( \phi \)-prime for every \( \phi \) and hence for some \( \phi \leq \phi_2 \). Conversely, let \( N \) be \( \phi \)-prime for some \( \phi \leq \phi_2 \). Then by Theorem 2, \( N \) is \( \phi_2 \)-prime. Let \( aA \leq N \) for \( a \in L, \ A \in M \). If \( aA \notin \phi_2(N) \), then as \( N \) is \( \phi_2 \)-prime, we have either \( A \not\leq N \) or \( a \leq (N : I_M) \). Next, assume that \( aA \leq \phi_2(N) \). If \( a(A \vee N) \notin \phi_2(N) \), then as \( a(A \vee N) \leq N \) and \( N \) is \( \phi_2 \)-prime, we have either \( (A \vee N) \leq N \) or \( a \leq (N : I_M) \) and hence either \( A \not\leq N \) or \( a \leq (N : I_M) \). Finally, if \( a(A \vee N) \leq \phi_2(N) \), then \( aN \leq (N : I_M)N \) which implies \( a \leq (N : I_M) \), by Lemma 2 and hence \( N \) is prime.

Definition 3. A proper element \( N \in M \) is said to be 2-potent prime if for all \( a \in L, \ A \in M \), \( aA \leq (N : I_M)N \) implies either \( a \leq (N : I_M) \) or \( A \not\leq N \).

Theorem 7. Let a proper element \( N \) of an \( L \)-module \( M \) be 2-potent prime. Then \( N \) is \( \phi \)-prime for some \( \phi \leq \phi_2 \) if and only if \( N \) is prime.

Proof. Assume that \( N \in M \) is a prime element. Then obviously, \( N \) is \( \phi \)-prime for every \( \phi \) and hence for some \( \phi \leq \phi_2 \). Conversely, let \( N \) be \( \phi \)-prime for some \( \phi \leq \phi_2 \). Then by Theorem 2, \( N \in M \) is \( \phi_2 \)-prime. Let \( aA \leq N \) for \( a \in L, \ A \in M \). If \( aA \notin (N : I_M)N \), then as \( N \) is \( \phi_2 \)-prime, we have either \( a \leq (N : I_M) \) or \( A \not\leq N \). If \( aA \leq (N : I_M)N \), then as \( N \) is 2-potent prime, we have either \( a \leq (N : I_M) \) or \( A \not\leq N \) and hence \( N \) is prime.

Now we define a \( n \)-potent prime element in an \( L \)-module \( M \) where \( n \geq 2 \).

Definition 4. Let \( n \geq 2 \) and \( n \in \mathbb{Z}_+ \). A proper element \( N \in M \) is said to be \( n \)-potent prime if for all \( a \in L, \ A \in M \), \( aA \leq (N : I_M)^{n-1}N \) implies either \( a \leq (N : I_M) \) or \( A \not\leq N \).
Theorem 8. A proper element $N$ of an $L$-module $M$ is $\phi$-prime for some $\phi \leq \phi_n$ where $n \geq 2$ if and only if $N$ is prime, provided $N$ is $k$-potent prime for some $k \leq n$.

Proof. Assume that $N \in M$ is a prime element. Then obviously, $N$ is $\phi$-prime for every $\phi$ and hence for some $\phi \leq \phi_n$ where $n \geq 2$. Conversely, let $N$ be $\phi$-prime for some $\phi \leq \phi_n$ where $n \geq 2$. Then by Theorem 2, $N \in M$ is $\phi_n$-prime. Let $aA \leq N$ for $a \in L$, $A \in M$. If $aA \not\leq \phi_k(N)$, then $aA \not\leq \phi_n(N)$ as $k \leq n$. Since $N$ is $\phi_n$-prime, we have either $a \leq (N : I_M)$ or $A \leq N$. If $aA \leq \phi_k(N)$, then as $N$ is $k$-potent prime, we have either $a \leq (N : I_M)$ or $A \leq N$ and hence $N$ is prime.

The following corollary is outcome of Theorems 5, 6 and 7.

Corollary 4. An almost prime element $N$ of an $L$-module $M$ is prime if one the following statements hold true:

(i) $M$ is torsion free and $O_M \neq N < I_M$ is a weak join principal element.

(ii) $N$ is a 2-potent prime element.

(iii) $L$ is a Noether PG-lattice, $M$ is a faithful multiplication PG-lattice with $I_M$ compact, $0 \neq (N : I_M) \in L$ satisfies the restricted cancellation law (RCL) and is a non-nilpotent element.

Theorem 9. Let a proper element $N$ of an $L$-module $M$ be $\phi$-prime. If $\phi(N)$ is prime, then $N$ is prime.

Proof. Let $aA \leq N$ for $a \in L$, $A \in M$. If $aA \not\leq \phi(N)$, then as $N$ is $\phi$-prime, we have either $a \leq (N : I_M)$ or $A \leq N$ and we are done. If $aA \leq \phi(N)$, then as $\phi(N)$ is prime, we have either $aI_M \leq \phi(N)$ or $A \leq \phi(N)$. This implies that either $aI_M \leq N$ or $A \leq N$ because $\phi(N) \leq N$. Hence $N$ is prime.

Theorem 10. Let a proper element $N$ of an $L$-module $M$ be $\phi$-prime. If $(N : I_M)N \not\leq \phi(N)$, then $N$ is prime.

Proof. Let $aA \leq N$ for $a \in L$, $A \in M$. If $aA \not\leq \phi(N)$, then as $N$ is $\phi$-prime, we have either $a \leq (N : I_M)$ or $A \leq N$. So assume that $aA \leq \phi(N)$. First suppose $aN \not\leq \phi(N)$. Then $aN_0 \not\leq \phi(N)$ for some $N_0 \leq N$ in $M$. Since $N$ is $\phi$-prime, $a(A \vee N_0) = aA \vee aN_0 \leq N$ and $a(A \vee N_0) \not\leq \phi(N)$, we have either $a \leq (N : I_M)$ or $(A \vee N_0) \leq N$ and hence either $a \leq (N : I_M)$ or $A \leq N$. Next, assume that $aN \leq \phi(N)$. If $(N : I_M)A \not\leq \phi(N)$, then $k_0A \not\leq \phi(N)$ for some $k_0 \leq (N : I_M)$ in $L$. Since $N$ is $\phi$-prime, $(a \vee k_0)A \leq N$ and $(a \vee k_0)A \not\leq \phi(N)$, we have either $a \leq (N : I_M)$ or $A \leq N$ and hence either $a \leq (N : I_M)$ or $A \leq N$. Now let $(N : I_M)A \leq \phi(N)$. By hypothesis, as $(N : I_M)N \not\leq \phi(N)$, there exist $k \leq (N : I_M)$ in $L$ and $N_0 \leq N$ in $M$ such that $kN_0 \not\leq \phi(N)$.

The consequences of Theorem 10 are presented in the following corollaries.
Corollary 5. If a proper element $N$ of a multiplication lattice $L$-module $M$ is $\phi$-prime but not prime, then $(N : I_M)^2I_M \leq \phi(N)$.

**Proof.** Since $M$ is a multiplication lattice $L$-module, by Proposition 3 of [10], we have $N = (N : I_M)I_M$. So $(N : I_M)^2I_M = (N : I_M)N \leq \phi(N)$ by Theorem 10.

Corollary 6. If a proper element $N$ of an $L$-module $M$ is weakly prime such that $(N : I_M)N \neq O_M$, then $N$ is prime.

**Proof.** The proof is obvious.

Corollary 7. If a proper element $N$ of an $L$-module $M$ is $\phi$-prime such that $\phi \leq \phi_3$, then $N$ is $\omega$-prime.

**Proof.** If $N$ is prime, then by Theorem 3, $N$ is $\omega$-prime. So assume that $N$ is not prime. Then by Theorem 10 and hypothesis, we get $(N : I_M)^3N \leq (N : I_M)N \leq \phi(N) \leq (N : I_M)^2N$ and so $\phi(N) = (N : I_M)^2N = (N : I_M)N$. Now consider $(N : I_M)^3N = ((N : I_M)(N : I_M)N = (N : I_M)^2N = (N : I_M)^2N = \phi(N)$ and so on. Hence $\phi(N) = (N : I_M)^{n-1}N$ for every $n \geq 2$. Consequently, $N$ is $n$-almost prime for every $n \geq 2$ and thus $N$ is $\omega$-prime by Corollary 3.

Corollary 8. If a proper element $N$ of a multiplication lattice $L$-module $M$ is $\phi$-prime but not prime, then $\sqrt{N : I_M} = \sqrt{\phi(N) : I_M}$.

**Proof.** By Corollary 5, we have $(N : I_M)^2I_M \leq \phi(N)$ which implies $(N : I_M) \leq \sqrt{\phi(N) : I_M}$. Hence $\sqrt{N : I_M} \leq \sqrt{\phi(N) : I_M} = \sqrt{\phi(N) : I_M}$, by property (p3) of radicals in [21]. Also, as $\phi(N) \leq N$, we have $\sqrt{\phi(N) : I_M} \leq \sqrt{N : I_M}$ and thus $\sqrt{N : I_M} = \sqrt{\phi(N) : I_M}$.

Corollary 9. If a proper element $N$ of a multiplication lattice $L$-module $M$ is $\phi$-prime, then either $\sqrt{\phi(N) : I_M} \leq (N : I_M)$ or $(N : I_M) \leq \sqrt{\phi(N) : I_M}$.

**Proof.** The proof is obvious.

Now we introduce the notion of $\phi$-primary element of an $L$-module $M$.

**Definition 5.** Let $\phi : M \rightarrow M$ be a function on an $L$-module $M$. A proper element $N \in M$ is said to be $\phi$-primary if for all $a \in L$, $A \in M$, $aA \leq N$ and $aA \notin \phi(N)$ implies either $A \leq N$ or $a^n \leq (N : I_M)$ for some $n \in \mathbb{Z}^+$.

Now if $\phi_\alpha : M \rightarrow M$ is a function on an $L$-module $M$, then $\phi_\alpha$-primary elements of $M$ are defined by following settings in the Definition 5 of a $\phi$-primary element.

- $\phi_0(N) = O_M$. Then $N \in M$ is called a weakly primary element.
- $\phi_2(N) = (N : I_M)N$. Then $N \in M$ is called a 2-almost primary element or a $\phi_2$-primary element or simply an almost primary element.
\[ \phi_{\omega}(N) = \bigwedge_{i=1}^{\infty} (N : I_M)^i N. \] Then \( N \subseteq M \) is called a \( \omega \)-primary element or \( \phi_{\omega} \)-primary element.

Clearly, every \( \phi \)-prime element of an \( L \)-module \( M \) is \( \phi \)-primary but the converse is not true as shown in the following example by taking \( \phi(N) = (N : I_M)N \) for convenience.

**Example 2.** Consider the lattice module as in Example 1. Let \( N \) be the cyclic submodule of \( M \) generated by \( T \). It is easy to see that the element \( N = < T > \) is almost primary (\( \phi_2 \)-primary) but \( N \) is not almost prime (\( \phi_2 \)-prime) because \( 2 < T > \subseteq N \), \( 2 < T > \nsubseteq N \) but \( N = < T > \).\( N \) and \( (2) \nsubseteq (N : I_M) = (4) \) where \( I_M = < T > \).

Clearly, every prime element of an \( L \)-module \( M \) is \( \phi \)-primary. But the converse is not true which is shown in the following example by taking \( \phi(N) = (N : I_M)N \) for convenience.

**Example 3.** Consider the lattice module as in Example 1. Let \( N \) be the cyclic submodule of \( M \) generated by \( \overline{0} \). It is easy to see that the element \( N = < \overline{0} > = O_M \) is almost primary (\( \phi_2 \)-primary) but \( N \) is not prime.

The analogous results (from the results of \( \phi \)-prime elements of \( M \)) for \( \phi \)-primary elements of \( M \) are stated below whose proofs being on similar arguments are omitted. We begin with the characterizations of a \( \phi \)-primary element of an \( L \)-module \( M \).

**Theorem 11.** Let \( M \) be a CG-lattice \( L \)-module, \( N \subseteq M \) be a proper element and \( \phi : M \rightarrow M \) be a function on \( M \). Then the following statements are equivalent:

1. \( N \) is a \( \phi \)-primary element of \( M \).
2. For every \( A \subseteq M \) such that \( A \nsubseteq N \), either \( (N : A) \subseteq \sqrt{N : I_M} \) or \( (N : A) = (\phi(N) : A) \).
3. For every \( L \) such that \( r \nsubseteq \sqrt{N : I_M} \), either \( (N : r) = N \) or \( (N : r) = (\phi(N) : r) \).
4. For every \( r \in L \), \( A \in M \), if \( rA \nsubseteq N \) and \( rA \nsubseteq \phi(N) \), then either \( r \leq \sqrt{N : I_M} \) or \( A \leq N \).

The following 2 corollaries are consequences of Theorem 11.

**Corollary 10.** Let \( M \) be a CG-lattice \( L \)-module and \( N \subseteq M \) be a proper element. Then the following statements are equivalent:

1. \( N \) is a weakly primary element of \( M \).
2. For every \( A \subseteq M \) such that \( A \nsubseteq N \), either \( (N : A) \subseteq \sqrt{N : I_M} \) or \( (N : A) = (O_M : A) \).
3. For every \( r \in L \) such that \( r \nsubseteq \sqrt{N : I_M} \), either \( (N : r) = N \) or \( (N : r) = (O_M : r) \).
For every \( r \in L_\ast \), \( A \in M_\ast \), if \( O_M \neq rA \leq N \), then either \( r \leq \sqrt{N : I_M} \) or \( A \leq N \).

**Corollary 11.** Let \( M \) be a CG-lattice \( L \)-module and \( N \in M \) be a proper element. Then the following statements are equivalent:

1. \( N \) is an almost primary element of \( M \).
2. For every \( A \in M \) such that \( A \leq N \), either \((N : A) = ((N : I_M)N : A) \) or \((N : A) \leq \sqrt{N : I_M} \).
3. For every \( r \in L \) such that \( r \leq \sqrt{N : I_M} \), either \((N : r) = ((N : I_M)N : r) \) or \((N : r) = N \).
4. For every \( r \in L_\ast \), \( A \in M_\ast \), if \( rA \leq N \) and \( rA \leq (N : I_M)N \), then either \( r \leq \sqrt{N : I_M} \) or \( A \leq N \).

To obtain the relation among primary, weakly primary, \( \omega \)-primary, \( n \)-almost primary (\( n \geq 2 \)) and almost primary elements of an \( L \)-module \( M \), we have the following result.

**Theorem 12.** Let \( \gamma_1, \gamma_2 : M \to M \) be functions on an \( L \)-module \( M \) such that \( \gamma_1 \leq \gamma_2 \). Then every proper \( \gamma_1 \)-primary element of \( M \) is \( \gamma_2 \)-primary.

**Theorem 13.** Let \( N \) be a proper element of an \( L \)-module \( M \). Then \( N \) is primary implies \( N \) is weakly primary, \( N \) is weakly primary implies \( N \) is \( \omega \)-primary, \( N \) is \( \omega \)-primary implies \( N \) is \( n \)-almost primary (\( n \geq 2 \)), \( N \) is \( n \)-almost primary (\( n \geq 2 \)) implies \( N \) is almost primary.

From the Theorem 13, we get the following characterization of a \( \omega \)-primary element of an \( L \)-module \( M \).

**Corollary 12.** Let \( N \in M \) be a proper element of an \( L \)-module \( M \). Then \( N \) is \( \omega \)-primary if and only if \( N \) is \( n \)-almost primary for every \( n \geq 2 \).

The following theorem gives the characterization of an \( n \)-almost primary element of an \( L \)-module \( M \).

**Theorem 14.** Let \( L \) be a Noether lattice, \( M \) be a torsion free Noetherian \( L \)-module and \( f \in L \) be the Jacobson radical. Then a proper element \( N \in M \) such that \((N : I_M) \leq f \) is \( n \)-almost primary for every \( n \geq 2 \) if and only if \( N \) is primary.

Clearly, every primary element of an \( L \)-module \( M \) is \( \phi \)-primary. But the converse is not true which is shown in the following example by taking \( \phi(N) = (N : I_M)N \) for convenience.

**Example 4.** If \( Z \) is the ring of integers, then \( Z_{30} \) is a \( Z \)-module. Assume that \((k)\) denotes the cyclic ideal of \( Z \) generated by \( k \in Z \) and \(<\overline{t}>\) denotes the cyclic submodule of \( Z \)-module \( Z_{30} \) where \( \overline{t} \in Z_{30} \). Suppose that \( L = L(Z) \) is the set of all ideals of \( Z \) and \( M = L(Z_{30}) \) is the set of all submodules of \( Z \)-module \( Z_{30} \). The multiplication between
elements of $L$ and $M$ is given by $(k_i) < t_j >= < k_i t_j >$ for every $(k_i) \in L$ and $< t_j > \in M$ where $k_i, t_j \in \mathbb{Z}$. Then $M$ is a lattice module over $L$. Let $N$ be the cyclic submodule of $M$ generated by $\mathcal{B}$. It is easy to see that $N = < \mathcal{B} >$ is almost primary ($\phi_2$-primary) while $N$ is not primary, since $(3) < \mathcal{B} >= N \text{ but } < \mathcal{B} > \not\subset N$ and $(3)^n \not\subset (N : I_M) = (6)$ for every $n \in \mathbb{Z}_+$ where $I_M = < T >$.

In the following successive nine theorems, we show under which condition(s) a $\phi$-primary element of an $L$-module $M$ is primary. Now we have a characterization of a $\phi$-primary element of an $L$-module $M$.

**Theorem 15.** Let $M$ be a torsion free $L$-module and $O_M \neq N < I_M$ be a weak join principal element of an $L$-module $M$. Then $N$ is $\phi$-primary for some $\phi \leq \phi_2$ if and only if $N$ is primary.

The following result shows that the Theorem 15 can also be achieved by changing the conditions on $O_M$ and $L$.

**Theorem 16.** Let $L$ be a Noether PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of $M$ such that $0 \neq (N : I_M) \in L$ satisfies the restricted cancellation law (RCL) and is a non-nilpotent element. Then $N$ is $\phi$-primary for some $\phi \leq \phi_2$ if and only if $N$ is primary.

Now we define a 2-potent primary element in an $L$-module $M$.

**Definition 6.** A proper element $N \in M$ is said to be 2-potent primary if for all $a \in L$, $A \in M$, $a A \leq (N : I_M)N$ implies either $A \leq N$ or $a^m \leq (N : I_M)$ for some $m \in \mathbb{Z}_+$.

**Theorem 17.** Let a proper element $N$ of an $L$-module $M$ be 2-potent primary. Then $N$ is $\phi$-primary for some $\phi \leq \phi_2$ if and only if $N$ is primary.

Clearly, every 2-potent prime element of an $L$-module $M$ is 2-potent primary.

**Theorem 18.** Let a proper element $N$ of an $L$-module $M$ be 2-potent primary. Then $N$ is $\phi$-primary for some $\phi \leq \phi_2$ if and only if $N$ is primary.

Now we define a $n$-potent primary element in an $L$-module $M$ where $n \geq 2$.

**Definition 7.** Let $n \geq 2$ and $n \in \mathbb{Z}_+$. A proper element $N \in M$ is said to be $n$-potent primary if for all $a \in L$, $A \in M$, $a A \leq (N : I_M)^n N$ implies either $A \leq N$ or $a^m \leq (N : I_M)$ for some $m \in \mathbb{Z}_+$.

**Theorem 19.** A proper element $N$ of an $L$-module $M$ is $\phi$-primary for some $\phi \leq \phi_n$ where $n \geq 2$ if and only if $N$ is primary, provided $N$ is $k$-potent primary for some $k \leq n$.

Clearly, every $n$-potent prime element of an $L$-module $M$ is $n$-potent primary.

**Theorem 20.** A proper element $N$ of an $L$-module $M$ is $\phi$-primary for some $\phi \leq \phi_n$ where $n \geq 2$ if and only if $N$ is primary, provided $N$ is $k$-potent prime for some $k \leq n$. 
The following corollary is outcome of Theorems 15, 16, 17 and 18.

**Corollary 13.** An almost primary element $N$ of an $L$-module $M$ is primary if one the following statements hold true:

(i) $M$ is torsion free and $O_M \neq N < I_M$ is a weak join principal element.

(ii) $N$ is a 2-potent primary element.

(iii) $N$ is a 2-potent prime element.

(iv) $L$ is a Noether PG-lattice, $M$ is a faithful multiplication PG-lattice with $I_M$ compact, $0 \neq (N : I_M) \in L$ satisfies the restricted cancellation law (RCL) and is a non-nilpotent element.

From the following examples, it is clear that, an almost primary element of an $L$ module $M$ need not be 2-potent prime and a 2-potent prime element of an $L$ module $M$ which is almost primary need not be prime.

**Example 5.** Consider the lattice module as in Example 4. Let $N$ be the cyclic submodule of $M$ generated by $6$. It is easy to see that the element $N = <6>$ is almost primary but not 2-potent prime.

**Example 6.** If $Z$ is the ring of integers, then $Z_8$ is a $Z-$module. Assume that $(k)$ denotes the cyclic ideal of $Z$ generated by $k \in Z$ and $<\bar{t}>$ denotes the cyclic submodule of $Z-$module $Z_8$ where $\bar{t} \in Z_8$. Suppose that $L = L(Z)$ is the set of all ideals of $Z$ and $M = L(Z_8)$ is the set of all submodules of $Z-$module $Z_8$. The multiplication between elements of $L$ and $M$ is given by $(k_i) <\bar{t}_j> = <\bar{k}_i\bar{t}_j>$ for every $(k_i) \in L$ and $<\bar{t}_j> \in M$ where $k_i, t_j \in Z$. Then $M$ is a lattice module over $L$. Let $N$ be the cyclic submodule of $M$ generated by $4$. It is easy to see that $N = <4>$ is almost primary ($\phi_2$-primary) and 2-potent prime but not prime.

**Theorem 21.** Let a proper element $N$ of an $L$-module $M$ be $\phi$-primary. If $\phi(N)$ is primary, then $N$ is primary.

**Theorem 22.** Let a proper element $N$ of an $L$-module $M$ be $\phi$-primary. If $(N : I_M)N \not\leq \phi(N)$, then $N$ is primary.

The consequences of Theorem 22 are presented in the form of following corollaries.

**Corollary 14.** If a proper element $N$ of a multiplication lattice $L$-module $M$ is $\phi$-primary but not primary, then $(N : I_M)^2I_M \leq \phi(N)$.

**Corollary 15.** If a proper element $N$ of an $L$-module $M$ is weakly primary such that $(N : I_M)N \neq O_M$, then $N$ is primary.

**Corollary 16.** If a proper element $N$ of an $L$-module $M$ is $\phi$-primary such that $\phi \leq \phi_3$, then $N$ is $\omega$-primary.
Corollary 17. If a proper element \( N \) of a multiplication lattice \( L \)-module \( M \) is \( \phi \)-primary but not primary, then \( \sqrt{N} : I_M = \sqrt{\phi(N)} : I_M \).

Corollary 18. If a proper element \( N \) of a multiplication lattice \( L \)-module \( M \) is \( \phi \)-primary, then either \( \sqrt{\phi(N)} : I_M \leq (N : I_M) \) or \( (N : I_M) \leq \sqrt{\phi(N)} : I_M \).

Theorem 23. Let a proper element \( N \) of an \( L \)-module \( M \) be \( \phi \)-primary. If \( (\sqrt{N} : I_M)N \neq \phi(N) \), then \( N \) is primary.

Proof. Just mimic the proof of Theorem 10.

Now, the interrelations among prime, primary, 2-absorbing and 2-absorbing primary elements of an \( L \)-module \( M \) are given in following theorems whose proofs being obvious are omitted.

Theorem 24. Every prime element of an \( L \)-module \( M \) is primary and 2-absorbing.

Theorem 25. If \( Q \) is a primary element of an \( L \)-module \( M \), then \( \sqrt{Q} : I_M \) is a prime element and hence a 2-absorbing element of \( L \). Also, it is a 2-absorbing primary element of \( L \).

Theorem 26. If \( Q \) is a 2-absorbing element of an \( L \)-module \( M \), then both \( \sqrt{Q} : I_M \) and \( (Q : I_M) \) are 2-absorbing elements of \( L \). Also, they are 2-absorbing primary elements of \( L \).

Theorem 27. Let \( L \) be a PG-lattice and \( M \) be a faithful multiplication PG-lattice \( L \)-module with \( I_M \) compact. If \( Q \) is a 2-absorbing primary element of \( M \), then \( (Q : I_M) \) is a 2-absorbing primary element of \( L \) and \( \sqrt{Q} : I_M \) is a 2-absorbing element of \( L \).

Proof. Let \( abc \leq (Q : I_M) \) for \( a, b, c \in L \). Then as \( abc \leq (Q : I_M) \) and \( Q \) is a 2-absorbing primary element of \( M \), we have, either \( ab \leq (Q : I_M) \) or \( ac \leq (\sqrt{Q} : I_M)I_M \) or \( b(cI_M) \leq (\sqrt{Q} : I_M)I_M \). Since \( I_M \) is compact, by Theorem 5 of [10], it follows that, either \( ab \leq (Q : I_M) \) or \( ac \leq \sqrt{Q} : I_M \) or \( bc \leq \sqrt{Q} : I_M \) and hence \( (Q : I_M) \) is a 2-absorbing primary element of \( L \). By Theorem 2.4 in [18], it follows that \( \sqrt{Q} : I_M \) is a 2-absorbing element of \( L \).

By relating the absorbing concepts with \( \phi \)-prime and \( \phi \)-primary elements of an \( L \)-module \( M \), we obtain the following results.

Theorem 28. Let a proper element \( N \) of an \( L \)-module \( M \) be \( \phi \)-prime. If \( (N : I_M)N \neq \phi(N) \), then \( N \) is primary and 2-absorbing. Also, both \( \sqrt{N} : I_M \) and \( (N : I_M) \) are 2-absorbing and hence 2-absorbing primary elements of \( L \).

Proof. The proof follows from Theorems 10, 24 and 26.

Clearly, every primary element of a multiplication \( L \)-module \( M \) is 2-absorbing primary.

Theorem 29. Let a proper element \( N \) of a multiplication \( L \)-module \( M \) be \( \phi \)-prime. If \( (N : I_M)N \neq \phi(N) \), then \( N \) is 2-absorbing primary. Also, then \( (N : I_M) \) is a 2-absorbing primary element of \( L \) provided \( M \) is a faithful PG-lattice with \( I_M \) compact and \( L \) as a PG-lattice. Further, \( \sqrt{N} : I_M \) is a 2-absorbing element of \( L \).
Proof. The proof follows from Theorems 10, 24 and 27.

**Theorem 30.** Let a proper element $N$ of a multiplication $L$-module $M$ be $\phi$-primary. If $(N : I_M)N \nleq \phi(N)$, then $N$ is 2-absorbing primary.

Proof. The proof follows from Theorem 22.

**Theorem 31.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let a proper element $N$ of an $L$-module $M$ be $\phi$-primary. If $(N : I_M)N \nleq \phi(N)$, then $(N : I_M)$ is a 2-absorbing primary element of $L$ and $\sqrt{N} : I_M$ is a 2-absorbing element of $L$.

Proof. The proof follows from Theorems 30 and 27.

The following results are obtained by relating the absorbing concepts with almost prime and almost primary elements of an $L$-module $M$.

**Theorem 32.** Let $M$ be a torsion free $L$-module and $O_M \neq N < I_M$ be a weak join principal element of $M$. If $N$ is almost prime, then $N$ is primary and 2-absorbing. Also, then both $\sqrt{N} : I_M$ and $(N : I_M)$ are 2-absorbing and hence 2-absorbing primary elements of $L$.

Proof. The proof follows from Theorems 5, 24 and 26.

**Theorem 33.** Let $M$ be a torsion free, multiplication $L$-module and $O_M \neq N < I_M$ be a weak join principal element of $M$. If $N$ is almost prime, then $N$ is 2-absorbing primary. Also, then $(N : I_M)$ is a 2-absorbing primary element of $L$ provided $M$ is a faithful PG-lattice with $I_M$ compact and $L$ as a PG-lattice. Further, $\sqrt{N} : I_M$ is a 2-absorbing element of $L$.

Proof. The proof follows from Theorems 34 and 27.

**Theorem 34.** Let $M$ be a torsion free, multiplication $L$-module and $O_M \neq N < I_M$ be a weak join principal element of $M$. If $N$ is almost primary, then $N$ is 2-absorbing primary.

Proof. The proof follows from Theorem 15.

**Theorem 35.** Let $M$ be a torsion free, faithful, multiplication PG-lattice $L$-module with $I_M$ compact and $L$ be a PG-lattice. Let $O_M \neq N < I_M$ be a weak join principal element of $M$. If $N$ is almost primary, then $(N : I_M)$ is a 2-absorbing primary element of $L$ and $\sqrt{N} : I_M$ is a 2-absorbing element of $L$.

Proof. The proof follows from Theorems 34 and 27.
3. Almost Prime and Almost Primary Elements in $M$

In this section, we will obtain some more results on an almost prime (respectively almost primary) element of an $L$-module $M$ by relating it with an idempotent element and a weakly prime (respectively weakly primary) element of an $L$-module $M$. Also, many characterizations of an almost prime and almost primary element of an $L$-module $M$ are obtained. Finally, we define $n$-potent prime (respectively $n$-potent primary) elements in $L$ and these notions are related with $n$-potent prime (respectively $n$-potent primary) elements in $M$ where $n \geq 2$.

Clearly, every almost prime element of an $L$-module $M$ is almost primary but the converse need not be true as seen in Example 2. It is easy to see that converse holds for radical elements of an $L$-module $M$. Every prime element of an $L$-module $M$ is almost prime and every primary element of an $L$-module $M$ is almost primary but their converses are not true as seen in Example 1 and Example 4, respectively. Also, every prime element of an $L$-module $M$ is almost primary.

According to Definition 2.6 of [22], an idempotent element of an $L$-module $M$ is defined in the following way.

**Definition 8.** A proper element $N$ of an $L$-module $M$ is said to be idempotent if $(N : I_M)N = N$.

Clearly, every idempotent element of an $L$-module $M$ is almost prime and hence almost primary. But an almost primary element of an $L$-module $M$ need not be idempotent as shown in the following example.

**Example 7.** Consider the lattice module as in Example 6. Let $N$ be the cyclic submodule of $M$ generated by $\mathfrak{A}$. It is easy to see that the element $N = < \mathfrak{A} >$ is almost primary but not idempotent.

**Theorem 36.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. For an idempotent element $N \in M$, $(\sqrt{(N : I_M)N : I_M})N = (N : I_M)N$.

**Proof.** As $N < I_M$ is idempotent, $N$ is almost prime ($\phi_2$ – prime). Since $M$ is a multiplication lattice $L$-module, we have $(N : I_M)^2I_M = (N : I_M)N$ which implies $(N : I_M) \leq \sqrt{(N : I_M)N : I_M}$. Thus $(N : I_M)N \leq \sqrt{(N : I_M)N : I_M}N$. Now to prove that $(\sqrt{(N : I_M)N : I_M})N \leq (N : I_M)N$, let $a \leq \sqrt{(N : I_M)N : I_M}$ for $a \in L$. If $a \leq (N : I_M)$, then we are done. So let $a \notin (N : I_M)$. Then as $N$ is $\phi_2$ – prime, by Theorem 1, we have either $(N : a) = N$ or $(N : a) = ((N : I_M)N : a)$. Let $(N : a) = N$ and $n$ be the least positive integer such that $a^n \leq ((N : I_M)N : I_M)$. If $n = 1$, then $aI_M \leq (N : I_M)N = (N : I_M)^2I_M$. As $I_M$ is compact, by Theorem 5 of [10], we have $a \leq (N : I_M)^2 \leq (N : I_M)$ which contradicts $a \notin (N : I_M)$. So assume that $n \geq 2$. Then $a^nI_M \leq (N : I_M)N \leq N$ with $a^nI_M \notin (N : I_M)N$ for every $k \leq (n - 1)$. Since
Lemma 4. \( a(a^{n-1}I_M) \leq N \), we have \( a^{n-1}I_M \leq (N : a) = N \) with \( a^{n-1}I_M \nless (N : I_M)N \). If \( n = 2 \), then \( aI_M \leq N \) which contradicts \( a \nless (N : I_M) \). If \( n \geq 3 \), then \( a(a^{n-2}I_M) \leq N \) but \( a(a^{n-2}I_M) \nless (N : I_M)N \). As \( N \) is almost prime, we have either \( a \leq (N : I_M) \) or \( a^{n-2}I_M \leq N \). As \( a \leq (N : I_M) \) is a contradiction, let \( a^{n-2}I_M \leq N \). Then \( a(a^{n-3}I_M) \leq N \) but \( a(a^{n-3}I_M) \nless (N : I_M)N \). As \( N \) is almost prime, we have either \( a \leq (N : I_M) \) or \( a^{n-3}I_M \leq N \). Continuing this process we conclude that \( a \leq (N : I_M) \) which contradicts \( a \nless (N : I_M) \). Hence we must have \( (N : a) = ((N : I_M)N : a) \). Then \( aN \leq a(N : a) = a((N : I_M)N : a) \leq (N : I_M)N \) which implies \( a \leq (N : I_M)N \) and so \( \sqrt{(N : I_M)N : I_M} \leq ((N : I_M)N : N) \). It follows that \( \sqrt{(N : I_M)N : I_M}N \leq (N : I_M)N \) and hence \( \sqrt{(N : I_M)N : I_M}N = (N : I_M)N \).

From following example, it is clear that an almost primary element of an \( L \)-module \( M \) need not be weakly primary.

Example 8. Consider the lattice module as in Example 4. Let \( N \) be the cyclic submodule of \( M \) generated by \( \mathcal{G} \). It is easy to see that the element \( N = \langle \mathcal{G} \rangle \) is almost primary (\( \phi_2 \)-primary) but not weakly primary.

Before obtaining the characterization of an almost primary element of an \( L \)-module \( M \) in terms of a weakly primary element of \( M \), we recall the definition of a local module \( M \). According to [1], an \( L \)-module \( M \) is said to be a local module if it has a unique maximal element.

Theorem 37. Let \( M \) be a local \( L \)-module with a unique maximal element \( Q \in M \) such that \( (Q : I_M)Q = O_M \). Then a proper element \( N \in M \) is almost primary if and only if \( N \) is weakly primary.

Proof. Assume that a proper element \( N \in M \) is almost primary. Then \( N \leq Q \). It follows that \( (N : I_M)N \leq (Q : I_M)Q = O_M \) and hence \( (N : I_M)N = O_M \). Let \( O_M \neq aA \leq N \) for \( a \in L, A \in M \). As \( aA \leq N \), \( aA \nless (N : I_M)N = O_M \) and \( N \) is almost primary, we have either \( a \leq N \) or \( a \leq \sqrt{N : I_M} \) and hence \( N \) is weakly primary. The converse is obvious from Theorem 13.

Now we prove the result required to show that if an element in \( M \) (or \( L \)) is almost primary, then its corresponding element in \( L \) (or \( M \)) is also almost primary.

Lemma 4. Let \( M \) be a torsion free multiplication lattice \( L \)-module and \( I_M \) be a weak join principal element of \( M \). Let \( N \) be a proper element of \( M \). Then \( a(N : I_M) = (aN : I_M) \) for \( a \in L \).

Proof. Since \( M \) is a multiplication lattice \( L \)-module, \( N = (N : I_M)I_M \). Then \( a(N : I_M)I_M = aN = (aN : I_M)I_M \) and so the result follows by Lemma 3.

Theorem 38. Let \( L \) be a PG-lattice and \( M \) be a faithful multiplication torsion free PG-lattice \( L \)-module with \( I_M \) compact. Let \( I_M \) be a weak join principal element and \( N \) be a proper element of \( M \). Then the following statements are equivalent:
1. \( N \) is an almost primary element of \( M \).

2. \((N : I_M)\) is an almost primary element of \( L \).

3. \( N = qI_M \) for some almost primary element \( q \in L \) which is maximal in the sense that if \( aI_M = N \), then \( a \leq q \) where \( a \in L \).

**Proof.** \( 1 \implies 2 \). Assume that \( N \) is an almost primary element of \( M \). Let \( ab \leq (N : I_M) \) and \( ab \not\leq (N : I_M)^2 \) for \( a, b \in L \). Then \( abI_M \leq N \). If \( abI_M \leq (N : I_M)N \), then by Lemma 4, we have \( ab \leq ((N : I_M)N : I_M) = (N : I_M)(N : I_M) \) which contradicts \( ab \not\leq (N : I_M)^2 \).

So let \( a(bI_M) \not\leq (N : I_M)N \). Then as \( N \) is almost primary, we have either \( a \leq \sqrt{N : I_M} \) or \( bI_M \leq N \) and thus \((N : I_M)\) is an almost primary element of \( L \).

\( 2 \implies 3 \). Assume that \((N : I_M) = q \) is an almost primary element of \( L \). Then \( qI_M \leq N \). Since \( M \) is a multiplication lattice module, \( N = aI_M \) for some \( a \in L \). So \( a \leq (N : I_M) = q \) and thus \( N = aI_M \leq qI_M \). Hence \( N = qI_M \) for some almost primary element \( q \in L \) which is maximal in the sense that if \( aI_M = N \), then \( a \leq q \).

\( 3 \implies 1 \). Suppose \( N = qI_M \) for some almost primary element \( q \in L \) which is maximal in the sense that if \( aI_M = N \), then \( a \leq q \) where \( a \in L \). Then \( q \leq (N : I_M) \). Now, let \( rX \leq N \), \( rX \not\leq (N : I_M)N \) and \( X \not\leq N \) for \( r \in L \), \( X \in M \). Since \( M \) is a multiplication lattice module, \( X = cI_M \) for some \( c \in L \). Then \( rc \leq (N : I_M) \leq q \), using maximality of \( q \) to \( N = (N : I_M)I_M \) (by Proposition 3 of [10]). If \( rc \leq q^2 \), then \( rX \leq qN \leq (N : I_M)N \), a contradiction. So \( rc \not\leq q^2 \). Also, \( c \not\leq q \) because if \( c \leq q \), then \( X \leq N \), a contradiction. Now, as \( rc \leq q \), \( rc \not\leq q^2 \), \( c \not\leq q \) and \( q \) is almost primary, we have, \( r \leq \sqrt{q} \) which implies \( r \leq \sqrt{N : I_M} \) and hence \( N \) is almost primary.

**Theorem 39.** Let \( L \) be a PG-lattice and \( M \) be a faithful multiplication torsion free PG-lattice \( L \)-module with \( I_M \) compact. Let \( I_M \) be a weak join principal element and \( N \) be a proper element in \( M \). Then the following statements are equivalent:

1. \( N \) is an almost primary element of \( M \).

2. \((N : I_M)\) is an almost primary element of \( L \).

3. \( N = qI_M \) for some almost primary element \( q \in L \).

**Proof.** \( 1 \implies 2 \) follows from \( 1 \implies 2 \) in the proof of Theorem 38.

\( 2 \implies 1 \). Assume that \((N : I_M)\) is an almost primary element of \( L \). Let \( rQ \leq N \) and \( rQ \not\leq (N : I_M)N \) for \( r \in L \), \( Q \in M \). Then \((rQ : I_M) \leq (N : I_M) \) and so by Lemma 4, we have \( rQ : I_M = (rQ : I_M) \leq (N : I_M) \). If \( rQ : I_M \not\leq (N : I_M)^2 = (N : I_M)(N : I_M) \), then \( rQ : I_M)I_M \leq (N : I_M)N \) which implies \( rQ \leq (N : I_M)N \), a contradiction. If \( rQ : I_M \not\leq (N : I_M)^2 \), then as \( rQ : I_M \leq (N : I_M) \) and \((N : I_M) \) is almost primary, we have either \( r \leq \sqrt{N : I_M} \) or \((Q : I_M) \leq (N : I_M) \) which implies either \( r \leq \sqrt{N : I_M} \) or \( Q \leq N \) and thus \( N \) is an almost primary element of \( M \).

\( 2 \implies 3 \). Suppose \((N : I_M)\) is an almost primary element of \( L \). Since \( M \) is a multiplication lattice \( L \)-module, \( N = (N : I_M)I_M \) and hence \( 3 \) holds.
Suppose $N = qI_M$ for some almost primary element $q \in L$. As $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M$. Since $I_M$ is compact, $\mathcal{R}$ holds by Theorem 5 of [10].

Now we relate the almost primary element $N \in M$ with $\text{rad}(N) \in M$, the radical of $N$. According to definition 3.1 in [17], the radical of a proper element $N$ in an $L$ module $M$ is defined as $\sqrt{\{P \in M \mid P \text{ is a prime element and } N \notin P\}}$ and is denoted as $\text{rad}(N)$. Using Theorem 3.6 of [17], we have the following interesting characterization of an almost primary element of $M$.

**Theorem 40.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication torsion free PG-lattice $L$-module with $I_M$ compact. Let $I_M \in M$ be a weak join principal element. Then a proper element $P \in M$ is almost primary ($\phi_2$–primary) if and only if whenever $N = aI_M$ and $K = bI_M$ in $M$ are such that $abI_M \leq P$ and $abI_M \notin (P : I_M)P$ then either $N \leq P$ or $K \leq \text{rad}(P)$ for $a, b \in L$.

**Proof.** Assume that $P \in M$ is almost primary. Let $N = aI_M$ and $K = bI_M$ in $M$ be such that $abI_M \leq P$ and $abI_M \notin (P : I_M)P$ for $a, b \in L$. Since $M$ is a multiplication lattice $L$-module, we have $a = (N : I_M)$ and $b = (K : I_M)$ and so $(K : I_M)(N : I_M)I_M = abI_M \leq P$ and $(K : I_M)(N : I_M)I_M \notin (P : I_M)P$. As $P \in M$ is almost primary, we have either $(N : I_M)I_M \leq P$ or $(K : I_M) \leq \sqrt{P} : I_M$ which implies either $N = (N : I_M)I_M \leq P$ or $K = (K : I_M)I_M \leq (\sqrt{P} : I_M)I_M = \text{rad}(P)$ by Theorem 3.6 of [17]. Conversely, assume that $abI_M \leq P$ and $abI_M \notin (P : I_M)P$ implies either $N \leq P$ or $K \leq \text{rad}(P)$ where $N = aI_M$ and $K = bI_M$ are in $M$ for $a, b \in L$. Let $rs \leq (P : I_M)$ and $rs \notin (P : I_M)^2$ where $S = rI_M$ and $Q = sI_M$ are in $M$ for $r, s \in L$. If $rsI_M \leq (P : I_M)P$, then since $M$ is a multiplication lattice $L$-module, we have $rsI_M \leq (P : I_M)^2I_M$. So by Theorem 5 of [10], we have $rs \leq (P : I_M)^2$, a contradiction. So let $rsI_M \notin (P : I_M)P$. Since $rsI_M \leq P$, by hypothesis, we have either $S \leq P$ or $Q \leq \text{rad}(P)$ which implies either $rI_M \leq P$ or $sI_M \leq \text{rad}(P) = (\sqrt{P} : I_M)I_M$, by Theorem 3.6 of [17]. So either $r \leq (P : I_M)$ or $s \leq \sqrt{P} : I_M$, by Theorem 5 of [10]. Thus $(P : I_M)$ is an almost primary element of $L$ and hence by Theorem 39, $P$ is an almost primary element of $M$.

Now we show that Lemma 4 can also be achieved by changing the conditions on $M$ and $I_M$.

**Lemma 5.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of $M$. Then $a(N : I_M) = (aN : I_M)$ for $a \in L$.

**Proof.** Since $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M$. Then $a(N : I_M)I_M = aN = (aN : I_M)I_M$ and we are done, by Theorem 5 of [10].

Lemma 5 is Lemma 3.5 of [22].

In view of Lemma 5, the Theorems 38, 39 and 40 can be restated in the following way.
Theorem 41. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$. Then the following statements are equivalent:

1. $N$ is an almost primary element of $M$.
2. $(N : I_M)$ is an almost primary element of $L$.
3. $N = qI_M$ for some almost primary element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leq q$ where $a \in L$.

Theorem 42. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$. Then the following statements are equivalent:

1. $N$ is an almost primary element of $M$.
2. $(N : I_M)$ is an almost primary element of $L$.
3. $N = qI_M$ for some almost primary element $q \in L$.

Theorem 43. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $P \in M$ is almost primary (i.e., $(P : I_M)P$) if and only if whenever $N = aI_M$ and $K = bI_M$ in $M$ are such that $abI_M \subseteq P$ and $abI_M \not\subseteq (P : I_M)P$ then either $N \subseteq P$ or $K \subseteq \text{rad}(P)$ for $a, b \in L$.

The following result is a consequence of the Theorem 42.

Corollary 19. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $N \in M$ is almost primary if and only if $(N : I_M)$ is an almost primary element of $L$.

The analogous results (from the results of almost primary elements of $M$) for almost prime elements of $M$ are as follows.

In Example 2.5 of [22], it is shown that an almost prime element of an $L$-module $M$ need not be weakly prime. The following characterization of an almost prime element of an $L$-module $M$ shows that under a certain condition, an almost prime element of an $L$-module $M$ is weakly prime.

Theorem 44. Let $M$ be a local $L$-module with a unique maximal element $Q \in M$ such that $(Q : I_M)Q = O_M$. Then a proper element $N \in M$ is almost prime if and only if $N$ is weakly prime.

Proof. Assume that a proper element $N \in M$ is almost prime. Then $N \subseteq Q$. It follows that $(N : I_M)N \subseteq (Q : I_M)Q = O_M$ and hence $(N : I_M)N = O_M$. Let $O_M \neq aA \subseteq N$ for $a \in L$, $A \in M$. As $aA \subseteq N$, $aA \not\subseteq (N : I_M)N = O_M$ and $N$ is almost prime, we have either $A \subseteq N$ or $a \leq (N : I_M)$ and hence $N$ is weakly prime. The converse is obvious from Theorem 3.

The following result shows that if an element in $M$ (or $L$) is almost prime, then its corresponding element in $L$ (or $M$) is also almost prime.
**Theorem 45.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication torsion free PG-lattice $L$-module with $I_M$ compact. Let $I_M$ be a weak join principal element and $N$ be a proper element of $M$. Then the following statements are equivalent:

1. $N$ is an almost prime element of $M$.
2. $(N : I_M)$ is an almost prime element of $L$.
3. $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leq q$ where $a \in L$.

**Proof.** 1$\implies$2. Assume that $N$ is an almost prime element of $M$. Let $ab \leq (N : I_M)$ and $ab \notin (N : I_M)^2$ for $a, b \in L$. Then $abI_M \leq N$. If $abI_M \leq (N : I_M)N$, then by Lemma 4, we have $ab \leq (N : I_M)(N : I_M) = (N : I_M)N$ which contradicts $ab \notin (N : I_M)^2$. So let $a(bI_M) \notin (N : I_M)N$. Then as $N$ is almost prime, we have either $a \leq (N : I_M)$ or $bI_M \leq N$ and thus $(N : I_M)$ is an almost prime element of $L$.

2$\implies$3. Assume that $(N : I_M) = q$ is an almost prime element of $L$. Then $qI_M \leq N$. Since $M$ is a multiplication lattice module, $N = aI_M$ for some $a \in L$. So $a \leq (N : I_M) = q$ and thus $N = aI_M \leq qI_M$. Hence $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leq q$.

3$\implies$1. Suppose $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leq q$ where $a \in L$. Then $q \leq (N : I_M)$. Now, let $rX \leq N$, $rX \notin (N : I_M)N$ and $X \notin N$ for $r \in L$, $X \in M$. Since $M$ is a multiplication lattice module, $X = cI_M$ for some $c \in L$. Then $rc \leq (N : I_M) \leq q$, using maximality of $q$ to $N = (N : I_M)I_M$ (by Proposition 3 of [10]). If $rc \leq q^2$, then $rX \leq qN \leq (N : I_M)N$, a contradiction. So $rc \notin q^2$. Also, $c \notin q$ because if $c \leq q$, then $X \leq N$, a contradiction. Now, as $rc \leq q$, $rc \notin q^2$, $c \notin q$ and $q$ is almost prime, we have, $r \leq q$ which implies $r \leq (N : I_M)$ and hence $N$ is almost prime.

**Theorem 46.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication torsion free PG-lattice $L$-module with $I_M$ compact. Let $I_M$ be a weak join principal element and $N$ be a proper element of $M$. Then the following statements are equivalent:

1. $N$ is an almost prime element of $M$.
2. $(N : I_M)$ is an almost prime element of $L$.
3. $N = qI_M$ for some almost prime element $q \in L$.

**Proof.** 1$\implies$2 follows from 1$\implies$2 in the proof of Theorem 45.

2$\implies$1. Assume that $(N : I_M)$ is an almost prime element of $L$. Let $rQ \leq N$ and $rQ \notin (N : I_M)N$ for $r \in L$, $Q \in M$. Then $(rQ : I_M) \leq (N : I_M)$ and so by Lemma 4, we have $r(Q : I_M) = (rQ : I_M) \leq (N : I_M)$. If $r(Q : I_M) \leq (N : I_M)^2 = ((N : I_M)N : I_M)$, then $r(Q : I_M)rI_M \leq (N : I_M)N$ which implies $rQ \leq (N : I_M)N$, a contradiction. If $r(Q : I_M) \notin (N : I_M)^2$, then as $r(Q : I_M) \leq (N : I_M)$ and $(N : I_M)$ is almost prime, we
have either $r \leq (N : I_M)$ or $(Q : I_M) \leq (N : I_M)$ which implies either $r \leq (N : I_M)$ or $Q \leq N$ and thus $N$ is an almost prime element of $M$.

2\rightarrow 3. Suppose $(N : I_M)$ is an almost prime element of $L$. Since $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M$ and hence 3 holds.

3\rightarrow 2. Suppose $N = qI_M$ for some almost prime element $q \in L$. As $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M$. Since $I_M$ is compact, 2 holds by Theorem 5 of [10].

The following result is another characterization of an almost prime element of an $L$-module $M$.

**Theorem 47.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication torsion free PG-lattice $L$-module with $I_M$ compact. Let $I_M$ be a weak join principal element. Then a proper element $P \in M$ is almost prime ($\phi_2$-prime) if and only if whenever $N = aI_M$ and $K = bI_M$ in $M$ are such that $abI_M \leq P$ and $abI_M \not\subseteq (P : I_M)P$ then either $N \leq P$ or $K \leq P$ for $a, b \in L$.

**Proof.** Assume that $P \in M$ is almost prime. Let $N = aI_M$ and $K = bI_M$ in $M$ be such that $abI_M \leq P$ and $abI_M \not\subseteq (P : I_M)P$ for $a, b \in L$. Since $M$ is a multiplication lattice $L$-module, we have $a = (N : I_M)$ and $b = (K : I_M)$ and so $(K : I_M)(N : I_M)I_M = abI_M \leq P$ and $(K : I_M)(N : I_M)I_M \not\subseteq (P : I_M)P$. As $P \in M$ is almost prime, we have either $(N : I_M)I_M \leq P$ or $(K : I_M)I_M \leq (P : I_M)$ which implies either $N = (N : I_M)I_M \leq P$ or $K = (K : I_M)I_M \leq P$. Conversely, assume that $abI_M \leq P$ and $abI_M \not\subseteq (P : I_M)P$ implies either $N \leq P$ or $K \leq P$ where $N = aI_M$ and $K = bI_M$ are in $M$ for $a, b \in L$. Let $rs \leq (P : I_M)$ and $rs \not\subseteq (P : I_M)^2$ where $S = rI_M$ and $Q = sI_M$ are in $M$ for $r, s \in L$. If $rsI_M \leq (P : I_M)P$, then since $M$ is a multiplication lattice $L$-module, we have $rsI_M \leq (P : I_M)^2I_M$. So by Theorem 5 of [10], we have $rs \leq (P : I_M)^2$, a contradiction. So let $rsI_M \not\subseteq (P : I_M)P$. Since $rsI_M \leq P$, by hypothesis, we have either $S \leq P$ or $Q \leq P$ which implies either $rI_M \leq P$ or $sI_M \leq P$ and so either $r \leq (P : I_M)$ or $s \leq (P : I_M)$. Thus $(P : I_M)$ is an almost prime element of $L$ and hence by Theorem 46, $P$ is an almost prime element of $M$.

In view of Lemma 5, the Theorems 45, 46 and 47 can be restated in the following way.

**Theorem 48.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$. Then the following statements are equivalent:

1. $N$ is an almost prime element of $M$.

2. $(N : I_M)$ is an almost prime element of $L$.

3. $N = qI_M$ for some almost prime element $q \in L$ which is maximal in the sense that if $aI_M = N$, then $a \leq q$ where $a \in L$.

**Theorem 49.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$. Then the following statements are equivalent:
Theorem 49 is Theorem 3.8 of [22].

Theorem 50. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $P \in M$ is almost prime if and only if whenever $N = aI_M$ and $K = bI_M$ in $M$ are such that $abI_M \leq P$ and $abI_M \not\in (P : I_M)P$ then either $N \leq P$ or $K \leq P$ for $a, b \in L$.

Theorem 50 is Theorem 3.14 of [22].

The following result is a consequence of the Theorem 49.

Corollary 20. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $N$ of an $L$-module $M$ is almost prime if and only if $(N : I_M)$ is an almost prime element of $L$.

According to [16], a proper element $q \in L$ is said to be 2-potent prime if for all $a, b \in L$, $ab \leq q^2$ implies either $a \leq q$ or $b \leq q$ and a proper element $q \in L$ is said to be 2-potent primary if for all $a, b \in L$, $ab \leq q^2$ implies either $a \leq q$ or $b \leq \sqrt{q}$.

In view of these definitions, we define $n$-potent prime and $n$-potent primary elements (where $n \geq 2$) in a multiplicative lattice $L$ in following way.

Definition 9. Let $n \geq 2$ and $n \in \mathbb{Z}_+$. A proper element $q \in L$ is said to be $n$-potent prime if for all $a, b \in L$, $ab \leq q^n$ implies either $a \leq q$ or $b \leq q$.

Definition 10. Let $n \geq 2$ and $n \in \mathbb{Z}_+$. A proper element $q \in L$ is said to be $n$-potent primary if for all $a, b \in L$, $ab \leq q^n$ implies either $a \leq q$ or $b \leq \sqrt[n]{q}$.

Now we show that if an element in $M$ is $n$-potent prime (respectively $n$-potent primary), then its corresponding element in $L$ is also $n$-potent prime (respectively $n$-potent primary) and vice-versa where $n \geq 2$.

Theorem 51. Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$ and $n \geq 2$. Then the following statements are equivalent:

1. $N$ is a $n$-potent prime element of $M$.
2. $(N : I_M)$ is a $n$-potent prime element of $L$.
3. $N = qI_M$ for some $n$-potent prime element $q \in L$. 

Proof. Since $M$ is a multiplication lattice $L$-module, by Proposition 3 of [10], we have $N = (N : I_M)I_M$.

(1) $\implies$ (2). Assume that $N$ is a $n$-potent prime element of $M$. Let $ab \leq (N : I_M)^n$ for $a, b \in L$. Then $a(bI_M) \leq (N : I_M)^{n-1}N$. As $N$ is $n$-potent prime, we have either $a \leq (N : I_M)$ or $bI_M \leq N$ and thus $(N : I_M)$ is a $n$-potent prime element of $L$.

(2) $\implies$ (1). Assume that $(N : I_M)$ is a $n$-potent prime element of $L$. Let $aX \leq (N : I_M)^{n-1}N$ for $a \in L$ and $X \in M$. $M$ being a multiplication lattice $L$-module, we have $X = cI_M$ for some $c \in L$. Clearly, $a(cI_M) \leq (N : I_M)^nI_M$. This implies that $ac \leq (N : I_M)^n$ by Theorem 5 of [10]. As $(N : I_M)$ is a $n$-potent prime, we have either $a \leq (N : I_M)$ or $c \leq (N : I_M)$ which implies either $a \leq (N : I_M)$ or $X = cI_M \leq (N : I_M)I_M = N$ and thus $N$ is a $n$-potent prime element of $M$.

(2) $\implies$ (3). Suppose $q = (N : I_M)$ is a $n$-potent prime element of $L$. Since $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M = qI_M$ and hence (3) holds.

(3) $\implies$ (2). Suppose $N = qI_M$ for some $n$-potent prime element $q \in L$. As $M$ is a multiplication lattice $L$-module, $N = (N : I_M)I_M$. Since $I_M$ is compact, (2) holds by Theorem 5 of [10].

**Theorem 52.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Let $N$ be a proper element of an $L$-module $M$ and $n \geq 2$. Then the following statements are equivalent:

1. $N$ is a $n$-potent primary element of $M$.
2. $(N : I_M)$ is a $n$-potent primary element of $L$.
3. $N = qI_M$ for some $n$-potent primary element $q \in L$.

**Proof.** Just mimic the proof of Theorem 51.

We conclude this paper with following 2 results which are outcomes of Theorems 51 and 52, respectively.

**Corollary 21.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $N$ of an $L$-module $M$ is 2-potent prime if and only if $(N : I_M)$ is a 2-potent prime element of $L$.

**Corollary 22.** Let $L$ be a PG-lattice and $M$ be a faithful multiplication PG-lattice $L$-module with $I_M$ compact. Then a proper element $N$ of an $L$-module $M$ is 2-potent primary if and only if $(N : I_M)$ is a 2-potent primary element of $L$.

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