Baer Elements In Lattice Modules

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Abstract. Let L be a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact and M be a module over L. In this paper we generalize the concepts of Baer elements, ∗-elements and closed elements and obtain the relation between ∗-elements and Baer elements and also closed elements and Baer elements. Some characterization are also obtain for closed elements of M and minimal prime elements of M.

2010 Mathematics Subject Classifications: 13A99

Key Words and Phrases: Prime element, primary element, lattice modules, Baer element, ∗-element, closed element.

1. Introduction

A multiplicative lattice L is a complete lattice provided with commutative, associative and join distributive multiplication in which the largest element 1 acts as a multiplicative identity. An element a ∈ L is called proper if a < 1. A proper element p of L is said to be prime if ab ≤ p implies a ≤ p or b ≤ p. If a ∈ L, b ∈ L, (a : b) is the join of all elements c in L such that cb ≤ a. A proper element p of L is said to be primary if ab ≤ p implies a ≤ p or bn ≤ p for some positive integer n. If a ∈ L then √a = ∨{x ∈ L | xn ≤ a, n ∈ Z+}. An element a ∈ L is called a radical element if a = √a. An element a ∈ L is called compact if a ≤ ∨bα a implies a ≤ bα1 ∨ bα2 ∨ ... ∨ bαn for some finite subset {α1, α2, ..., αn}. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact and every finite product of compact elements is compact. We shall denote by Lc the set compact elements of L. A nonempty subset F of Lc is called a filter of Lc if the following conditions are satisfied,

(i) x, y ∈ F implies xy ∈ F
(ii) x ∈ F, x ≤ y implies y ∈ F.

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Let \( F(L) \) denote the set of all filters of \( L \). For a nonempty subset \( \{F_o\} \subseteq F(L) \), define 
\[
\bigcup F_o = \{ x \in L \mid x \geq f_1 f_2 \cdots f_n \in F_o, \text{ for some } i = 1, 2, \ldots, n \}.
\] Then it is observed that, 
\[
F(L) = (F(L), \cup, \cap, \emptyset)
\] is a complete distributive lattice with \( \cup \) as the supremum and the set theroretic \( \cap \) as the infimum. For \( a \in L \), the smallest filter containing \( a \) is denoted by \( [a] \) and it is given by \( [a] = \{ x \in L^{\ast} \mid x \geq a^n \text{ for some nonnegative integer } n \} \). For a filter \( F \in F(L) \) we denote \( \emptyset \) by \( \bigvee \{ x \in L \mid x s = 0, \text{ for } s \in F \} \).

Let \( M \) be a complete lattice and \( L \) be a multiplicative lattice. Then \( M \) is called \( L \)-module or module over \( L \) if there is a multiplication between elements of \( L \) and \( M \) written as \( a b \) where \( a \in L \) and \( b \in M \) satisfies the following properties,

\[
\begin{align*}
(i) \quad & (\forall_{a} a A = \forall_{a} a A \quad \forall a, A \in M \\
(ii) \quad & (\forall_{a} a A) = (\forall_{a} a A) \quad \forall a, A \in M \\
(iii) \quad & (a b) A = a (b A) \quad \forall a, b, A \in L, A \in M \\
(iv) \quad & 1 B = B \\
(v) \quad & 0 B = 0 M \quad \forall a, a, b \in L \text{ and } A, A_{a} \in M, \text{ where } 1 \text{ is the supremum of } L \text{ and } 0 \text{ is the infimum of } L. \text{ We denote by } 0_{M} \text{ and } I_{M} \text{ the least element and the greatest element of } M. \text{ Elements of } L \text{ will generally be denoted by } a, b, c, \ldots \text{ and elements of } M \text{ will generally be denoted by } a, b, c, \ldots.
\end{align*}
\]

Let \( M \) be a \( L \)-module. If \( N \in M \) and \( a \in L \) then \( (N : a) = \bigvee \{ X \in M \mid a X \leq N \} \). If \( A, B \in M \), then \( (A : B) = \bigvee \{ X \in L \mid X B \leq A \} \). An \( L \)-module \( M \) is called a multiplication \( L \)-module if for every element \( N \in M \) there exists an element \( a \in L \) such that \( N = a I_{M} \) see [2]. In this paper a lattice module \( M \) will be a multiplication lattice module, which is compactly generated with the largest element \( I_{M} \) compact. A proper element \( N \) of \( M \) is said to be prime if \( a X \leq N \) implies \( X \leq N \) or \( a I_{M} \leq N \) that is \( a \leq (N : I_{M}) \) for every \( a \in L, X \in M \). If \( N \) is a prime element of \( M \) then \( (N : I_{M}) \) is prime element of \( L \) [4]. An element \( N < I_{M} \) in \( M \) is said to be primary if \( a X \leq N \) implies \( X \leq N \) or \( a^n I_{M} \leq N \) that is \( a^n \leq (N : I_{M}) \) for some integer \( n \). An element \( N \) of \( M \) is called a radical element if \( (N : I_{M}) = \sqrt{(N : I_{M})} \). If \( a N = 0_{M} \) implies \( a = 0 \) or \( N = 0_{M} \) for any \( a \in L \) and \( N \in M \) then \( M \) is called a torsion free \( L \)-module.

2. Residuation properties

We state some elementary properties of residuation in the following theorem.

**Theorem 1.** Let \( L \) be a multiplicative lattice and \( M \) be a multiplication lattice module over \( L \). For \( x, y \in L \) and \( Z, A, B \in M \), where \( (0_{M} : I_{M}) \) is a radical element. We have the following identities,

\[
\begin{align*}
(i) \quad & x \leq y \text{ implies } (0_{M} : y) \leq (0_{M} : x) \text{ and } 0_{M} : (0_{M} : x) \leq 0_{M} : (0_{M} : y) \\
(ii) \quad & x \leq 0_{M} : (0_{M} : x) \\
(iii) \quad & 0_{M} : [0_{M} : (0_{M} : x)] = (0_{M} : x)
\end{align*}
\]
(iv) \((0_M : x) = (0_m : x^n)\) for every \(n \in \mathbb{Z}_+\)

(v) \(0_M : (0_M : x) \land 0_M : (0_M : y) = 0_M : (0_M : xy) = 0_M : (x \land y)\)

(vi) \((0_M : a) = 0_M \implies (0_M : x \land 0_M : y) = 0_M : (x \land y) = 0_M : xy\)

(vii) \(x \lor y = 1 \implies (0_M : x) \lor (0_M : y) = 0_M : (x \lor y) = 0_M : x y\)

(viii) For \(Z \in \mathbb{Z}, Z \leq 0_M : (0_M : Z)\)

(ix) \(A \leq B \implies (0_M : B) \leq (0_M : A)\)

(x) \(0_M : (0_M : A) = 0_M : A\)

(xi) \(0_M : x I_M = 0_M : x^n I_M\) for some positive integer \(n\).

We define, \(0_{FM} = \lor \{X \in M_\ast \mid sX = 0_M \text{ for some } s \in F\}\), where \(M_\ast\) is the set of compact elements of \(M\). The proofs of the following theorems are simple.

**Theorem 2.** Let \(F \subseteq L\) be a filter of \(F(L_\ast)\) and let \(X\) be a compact element of \(M\). Then \(X \leq 0_{FM}\) if and only if \(sX = 0_M\) for some \(s \in F\).

**Theorem 3.** For \(F \in F(L_\ast)\), \(0_{FM} = \lor \{(0_M : x) \mid x \in F\}\).

**Theorem 4.** For \(F_1, F_2 \in F(L_\ast)\)

(i) \(F_1 \subseteq F_2 \implies 0_{F_1,M} \leq 0_{F_2,M}\).

(ii) \(0_{F_1,M} \land 0_{F_2,M} = 0_{(F_1 \cap F_2)M}\)

3. **Baer Elements**

A study of Baer elements, \(\ast\)-elements and closed elements carried out by D D Anderson, et al. [1]. We generalize these concepts for lattice modules.

**Definition 1.** An element \(A \in M\) is said to be Baer element if for \(x \in L_\ast\), \(x I_M \leq A\) implies \(0_M : (0_M : x I_M) \leq A\).

**Definition 2.** An element \(A\) of \(M\) is said to be \(\ast\)-element if \(A = 0_{FM}\) for some filter \(F \in F(L_\ast)\) such that zero does not belong to \(F\).

**Definition 3.** An element \(A\) of \(M\) is said to be closed element if \(A = 0_M : (0_M : A)\).

The next result establishes the relation between closed element and Baer element.

**Theorem 5.** Every closed element is a Baer element.

**Proof.** Let \(A\) be a closed element of \(M\) and \(x\) be a compact element of \(L_\ast\) such that \(x I_M \leq A\). Then \(0_M : (0_M : x I_M) \leq 0_M : (0_M : A) = A\) as \(A\) is a closed. This shows that \(A\) is a Baer element. \(\Box\)
Definition 4. An element $P$ of $M$ is called a minimal prime element over $A \in M$ if $A \leq P$ and there is no other prime element $Q$ of $M$ such that $A \leq Q < P$.

The following result gives the characterization of a minimal prime element over an element.

Theorem 6. Let $a$ be a proper element of $L$ and $P$ be a prime element of $M$ with $a I_M \leq P$. Then the following statements are equivalent,

(i) $P$ is minimal prime element over $a I_M$.

(ii) For each compact element $x$ in $L$, $x I_M \leq P$, there is compact element $y$ in $L$ such that $y I_M \not\leq P$ and $x^n y I_M \leq a I_M = \ast$ for some positive integer $n$.

Proof. (i) $\Rightarrow$ (ii)

Let $P$ be a minimal prime over $a I_M$ and suppose $x I_M \leq P$. Let

$$S = \{x^n y \mid y \not\in (P : I_M) \text{ and } n \text{ is a positive integer}\}.$$ 

It is clear that, $S$ is a multiplicatively closed set. Suppose $x^n y \not\in a I_M$ for any integer $n$ and for any $y I_M \not\leq P$, where $y$ is compact in $L$. By the separation lemma (see [5]), there is a prime element $(Q : I_M)$ of $L$ such that $(P : I_M) \leq (Q : I_M)$ and $t \not\in (Q : I_M)$ for all $t \in S$. Then we have $(Q : I_M) \leq (P : I_M)$ since otherwise $x^n (Q : I_M) \in S$ and $x^n (Q : I_M) \not\in (Q : I_M)$ a contradiction. Hence $(P : I_M) = (Q : I_M)$. It follows that $P = Q$ (see [3]). But then for $t \in S$, $t \leq x \leq (P : I_M) = (Q : I_M)$ a contradiction.

(ii) $\Rightarrow$ (i)

Suppose for any $x$ in $L$, $x I_M \leq P$, there is $y$ in $L$ such that $y I_M \not\leq P$ and $x^n y I_M \not\leq a I_M$ for some positive integer $n$. Also suppose that there is a prime element $Q$ of $M$ with $a I_M \leq Q < P$. Choose, $x I_M \leq P$ and $x I_M \not\in Q$. By hypothesise, there is a compact element $y$ in $L$ such that $y I_M \not\leq P$ and integer $n$ such that $x^n y I_M \not\leq a I_M \leq Q$. As $x I_M \not\in Q$, $x \not\in (Q : I_M)$. Since $Q$ is a prime element of $M$, $(Q : I_M)$ is also prime element of $L$ (see [4]). Hence $x^n \not\in (Q : I_M)$. Thus, $x^n \not\in (Q : I_M)$ and $y \not\in (Q : I_M)$ where $(Q : I_M)$ is a prime element of $L$, which is a contradiction.

In the next result, we prove the important property of a minimal prime element.

Theorem 7. Let $M$ be a lattice module. Every minimal prime element of $M$ is a $\ast$-element where $0_{FM}$ is prime element.

Proof. Let $p$ be a minimal prime element of $M$. Define the set $F = \{x \in L \mid x I_M \not\in P\}$. We first show that $F$ is a filter of $F(L)$. Let $x$ and $y$ be compact element of $L$ such that $x, y \in F$. So $x I_M \not\in P$ and $y I_M \not\in P$. As $P$ is prime, $x y I_M \not\in P$. This shows that $x y \in F$. Now let $x \in F$ and $x \leq y$. Hence $x I_M \not\in P$ implies $y I_M \not\in P$ and $y \in F$. If $0 \in F$ then we have $0 I_M \not\in P$ that is $0 M \not\leq P$ a contradiction. Thus $F \in F(L)$ and $0 \not\in F$. Now we show that $P = 0_{FM}$. Let $x$ be a compact element of $L$ such that $x I_M \leq P$. By Theorem 6 it follows that there exist a compact element $y \in L$ such that $y I_M \not\leq P$ and $x^n y I_M = 0_M$ for some positive integer $n$. We have $y \in F$ and $x^n I_M \leq 0_{FM}$. As $0_{FM}$ is prime element, so $x I_M \leq 0_{FM}$ implies $P \leq 0_{FM}$. Now let $x$ be a
compact element of L such that \( xI_M \leq 0_{FM} \). Then by Theorem 2, \( rxI_M = 0_M \) for some \( r \in F \).
So we have \( r \neq I_M \leq P \) and \( rI_M \neq P \). As P is prime, \( xI_M \leq P \) and \( 0_{FM} \leq P \) which shows that \( P = 0_{FM} \). Thus every minimal prime element of M is \(*\)-element.

The relation between \(*\)-element and Baer element is proved in the next result.

**Theorem 8.** Each \(*\)-element of M is a Baer element.

**Proof.** Suppose an element \( A \) of M is \(*\)-element. Hence \( A = 0_{FM} \) for some filter \( F \in F(L_s) \) such that \( 0 \notin F \). Let \( x \in L_s \) such that \( xI_M \leq A \). Then we have \( rI_M = 0_M \) that is \( xI_M \leq (0_M : r) \) for some \( r \in F \) by Theorem 2. Therefore by (i) and (iii) of Theorem 1 we get

\[
0_M : (0_M : xI_M) \leq 0_M : [0_M : (0_M : r)] = (0_M : r).
\]

Hence by Theorem 3, \( 0_M : (0_M : xI_M) \leq \bigvee_{s \in F} (0_M : s) = 0_{FM} = A \). This shows that \( A \) is a Baer element.

The next result we prove the existence of closed and Baer elements.

**Theorem 9.** Let M be multiplication lattice module. For any \( x \in L_s(0_M : x) \) is both Baer and closed element.

**Proof.** For an element \( x \in L_s \), let \( xI_M \leq (0_M : x) \), then

\[
0_M : (0_M : xI_M) \leq 0_M : [0_M : (0_M : x)] = (0_M : x)
\]

by (i) and (iii) of Theorem 1. Thus \( (0_M : x) \) is a Baer element. Again from (iii) of Theorem 1, \( (0_M : x) = 0_M : (0_M : (0_M : x)) \). This shows that \( (0_M : x) \) is a closed element.

In the following theorem we prove the characterization of closed element in terms of Baer element.

**Theorem 10.** For \( a \in L_s \), a \( I_M \) is closed if and only if \( aI_M = 0_{FM} \) is a Baer element.

**Proof.** Let \( L_s \) be the set of all compact element of L and a \( I_M \) be a Baer element of M. We show that \( aI_M = 0_M : (0_M : aI_M) \). As a \( I_M \leq aI_M \), we have \( 0_M : (0_M : aI_M) \leq aI_M \). But \( aI_M : (0_M : aI_M) \leq 0_M \) implies \( aI_M \leq 0_M : (0_M : aI_M) \). Therefore \( 0_M : (0_M : aI_M) = aI_M \). Thus \( aI_M \) is closed. The converse is proved in Theorem 5.

**Theorem 11.** For a nonzero compact element \( a \) in \( L_s \), \( 0_M : a = 0_{(a)} \).

**Proof.** We note that \( F = [a] = \{ z \in L_s \mid z \geq a^n \text{ for some } n \in Z_+ \} \in F(L_s) \) and \( 0_{FM} = \bigvee [X \in M_s \mid sX = 0_M \text{ for some } s \in F] \). Now let \( z \) be compact element of L such that \( z \in F \cap \{0\} \). Then \( z \in F \) and \( z = 0 \). As \( z \in F, z \geq a^n \) for some \( n \in Z_+ \). Hence \( a = \sqrt{z} = 0 \) which shows that \( a = 0 \). This contradiction implies that \( 0 \notin F \). Now we show that \( 0_M : a = 0_{FM} \). As \( a \) is a compact element in \( L_s, a \in F \). So we have \( 0_M : a \leq 0_{FM} = \bigvee \{ (0_M : x) \mid x \in F \} \). Let \( Z \) be a compact element in \( M \) and \( Z \leq 0_{FM} \). Then by Theorem 2 \( sZ = 0_M \) for some \( s \in F \). So \( s \geq a^n = \) for some \( n \in Z_+ \). We note that \( 0_M : a^n = 0_M : a \). Consequently, we have \( a^n Z \leq sZ = 0_M \). This implies that \( Z \leq (0_M : a^n) = (0_M : a) \). Consequently, \( 0_F \leq (0_M : a) \) and \( (0_M : a) = 0_F \).
Theorem 12. Suppose L has no divisors of zero then the element 0_M is always a Baer, closed and *-element whereas 1_M is Baer and closed.

Proof. Let x be a nonzero element of L. From Theorem 9, for any x ∈ L, 0_M : x is both Baer and closed and by Theorem 11 for a nonzero compact element x of L, 0_M : x = 0_{(x)}. To show that 0_M a is Baer element, take x ∈ L, such that xI_M ≤ 0_M. We have

0_M : (0_M : xI_M) ≤ O_M : (0_M : 0_M) = 0_M.

Hence 0_M is a Baer element. As 0_M = 0_M : (0_M : 0_M), 0_M is closed. Every Baer element is a *-element. To show that 1_M is a Baer element, take any x ∈ L, such that xI_M ≤ 1_M. We have 0_M : (0_M : xI_M) = 0_M : [v{a ∈ L | axI_M = 0_M}] = 0_M : 0 = 1_M. So 1_M is a Baer element. Now 0_M : (0_M : 1_M) = 0_M : [v{a ∈ L | aI_M = 0_M}] = 1_M and 1_M is closed.

Remark 1. For defining the *-element, the condition 0 ∉ F is necessary.

Suppose if possible x is a *-element. Hence X = 0_F M, for some filter F such that 0 ∉ F. Then we have X = v{(0_M : r) | r ∈ F}. Now 0_M : 0 = v{A ∈ M | 0 = 0_M} = 1_M. Thus only 1_M will be a *-element. Hence, for defining a *-element we take F such that 0 ∉ F.

Theorem 13. If {A_α}_α is a family of Baer elements then \( A_\cap \alpha \) is a Baer element.

Proof. Let x ∈ L, such that xI_M ≤ A_α. Then for each α, xI_M ≤ A_α. As each A_α is a Baer element, 0_M : (0_M : xI_M) ≤ A_α. Hence 0_M : (0_M : xI_M) ≤ A_\cap \alpha. Thus A_\cap \alpha is a Baer element.

The next result we prove the relation between minimal prime element and Baer element.

Theorem 14. If A is a meet of minimal prime elements then A is a Baer element.

Proof. From Theorem 7, every minimal prime element of M is a *-element and by Theorem 8, each *-element of M is a Baer element. From these two results, every minimal prime element is a Baer element. So meet of all minimal prime elements is a Baer element, by Theorem 13.

Theorem 15. If {A_α}_α is a family of closed elements then A_\cap \α is a closed element.

Proof. We have A_\cap \alpha ≤ A_\alpha for each α. As each A_\alpha is a closed element we have 0_M : [0_M : (A_\cap \alpha)] ≤ 0_M : (0_M : A_\alpha) = A_\alpha. This gives 0_M : [0_M : (A_\cap \alpha)] ≤ A_\alpha. Now let Z be an element of M such that Z ≤ A_\alpha. Then we have Z ≤ 0_M : (0_M : Z) ≤ 0_M : (0_M : A_\alpha), by (ix) of Theorem 1. This gives A_\alpha ≤ 0_M : [0_M : (A_\cap \alpha)]. Thus we get 0_M : [0_M : (A_\cap \alpha)] = A_\alpha.

Here is an important property of largest element of M which is compact.

Theorem 16. 1_M is never a *-element where 1_M is compact and M is torsion free L-module.
Proof. Suppose that $1_M$ is a $*$-element. Then there exist some filter $F \in F(L_n)$ such that $1_M = 0_{FM}$, where $0 \notin F$. Then as $1_M$ is compact and $1_M = 0_{FM} = \bigvee \{0_M : x \in F\}$, $1_M = (0_M : x_1) \lor (0_M : x_2) \lor \ldots \lor (0_M : x_n)$ for some $x_1, x_2, \ldots, x_n \in F$. Consequently, as $1_M$ is closed,

$$1_M = 0_M : (0_M : 1_M) = 0_M : [0_M : ((0_M : x_1) \lor (0_M : x_2) \lor \ldots \lor (0_M : x_n))] = 0_M : [0_M : (0_M : x_1) \land 0_M : (0_M : x_2) \land \ldots \land 0_M : (0_M : x_n)].$$

Therefore $1_M = 0_M : [0_M : (0_M : (x_1 x_2 \ldots x_n))] = 0_M : (x_1 x_2 \ldots x_n)$, by (iii) and (v) of Theorem 1. This implies that $x_1 x_2 \ldots x_n = 0$. Since $x_1, x_2, \ldots, x_n$ are in $F$ we have $0 = x_1 x_2 \ldots x_n \in F$. Which is a contradiction as $0 \notin F$. \hfill \square

The next result we prove the characterization of a Baer element.

**Theorem 17.** The following statements are equivalent,

(i) An element $A \in M$ is a Baer element.

(ii) For any element $x, y \in L$ such that $x$ is compact $0_M : xI_M = 0_M : yI_M$ and $xI_M \leq A$ implies $yI_M \leq A$.

(iii) For any element $x, y \in L, 0_M : x = 0_M : y$ and $xI_M \leq A$ implies $yI_M \leq A$.

Proof. (i) $\Rightarrow$ (ii)
Assume that $A$ is a Baer element of $M$. Let $x, y \in L$ be such that $x$ is compact, $xI_M \leq A$, and $0_M : xI_M = 0_M : yI_M$. Then by Theorem 1, $yI_M \leq 0_M : (0_M : yI_M) = 0_M : (0_M : xI_M) \leq A$, since $A$ is a Baer element.

(ii) $\Rightarrow$ (iii)
Obvious.

(iii) $\Rightarrow$ (i)
Assume that for any element $x, y \in L, 0_M : xI_M = 0_M : yI_M$ and $xI_M \leq A$ implies $yI_M \leq A$. We show that $A \in M$ is a Baer element. Let $x \in L$ be such that $xI_M \leq A$. We have $0_M : xI_M = 0_M : [0_M : (0_M : xI_M)]$. Hence by (iii), we have $0_M : (0_M : xI_M) \leq A$. Hence, $A$ is a Baer element. \hfill \square

In the following theorem we prove the relation between Baer element of a lattice module and radical of a multiplicative lattice.

**Theorem 18.** If $A$ is a Baer element of $M$ then $A : I_M$ is a radical element.

Proof. Let $A$ be Baer element of a lattice module $M$. We show that $(A : I_M) = \sqrt{(A : I_M)}$. Assume that $x$ is a compact element such that $x^n I_M \leq A$ for some positive integer $n$. We have $0_M : xI_M = 0_M : x^n I_M$, by (xii) of Theorem 1 and hence by above theorem $xI_M \leq A$ that is $x \leq (A : I_M)$. Hence $\sqrt{(A : I_M)} \leq (A : I_M)$ and we have $\sqrt{(A : I_M)} = (A : I_M)$ i.e. $(A : I_M)$ is a radical element. \hfill \square

**Theorem 19.** If $A$ is a Baer element then every minimal prime element over $A$ is a Baer element.
Proof. Let A be a Baer element and P be a minimal prime in M over A. Assume that 0_M : x = 0_M : z for some x, z ∈ L such that x is compact and xI_M ⊆ P. There exists a compact element y ∈ L such that yI_M ∉ P and x^n yI_M ⊆ A ⊆ P for some positive integer n, by Theorem 14. Note that 0_M : yx = (0_M : x) : y = (0_M : x^n) : y = 0_M : x^n y = 0_M : yx^n = 0_M : yz. As A is a Baer element. By Theorem 17, xyI_M ⊆ A implies yzI_M ⊆ A ⊆ P. Hence zI_M ⊆ P as P is prime. So again by Theorem 17, P is a Baer element.

The characterization of minimal prime element of M is proved in the next theorem.

Theorem 20. Let L be a lattice module and P be a prime element of M. Then P is a minimal prime element if and only if for x ∈ L, P contains precisely one of xI_M and 0_M : x.

Proof. If part: Assume that for x ∈ L, P contains precisely one of xI_M and 0_M : x. First assume that P contains xI_M. But 0_M : x ∉ P. Therefore there exists a compact element y in L such that yI_M ⊆ 0_M : x but yI_M ∉ P. Thus xyI_M ⊆ P. This shows that for each compact element x in L, xI_M ⊆ P, there exist a compact element y in L such that yI_M ∉ P and xyI_M ⊆ 0_M. By Theorem 6, it follows that P is a minimal prime element of M. Next assume that 0_M : x ∉ P but xI_M ⊆ P. Let z be a compact element of L such that zI_M ⊆ (0_M : x) ⊆ P. But xI_M ∉ P and xzI_M ⊆ 0_M. Consequently, by Theorem 6 P is a minimal prime element. Thus the condition is sufficient.

Only if part: Assume that P is a minimal prime element of M. Let x be a compact element of L. Suppose if possible xI_M ⊆ P. Then by Theorem 6, there exist a compact element y in L such that yI_M ∉ P and x^n yI_M = 0_M for some positive integer n. Consequently, yI_M ⊆ 0_M : x^n = 0_M : x. This implies that 0_M : x ∉ P. Now suppose if possible xI_M ∉ P and 0_M : x ∉ P. Then there exist a compact element y in L such that yI_M ⊆ 0_M : x but yI_M ∉ P. Hence we have xyI_M ⊆ 0_M and so xyI_M ⊆ P. But xI_M ∉ P and yI_M ∉ P which contradicts the fact that P is prime element of M. This shows that P contains precisely one of xI_M and (0_M : x).

The relation between ∗-element of M and a minimal prime element over it is established in the next theorem.

Theorem 21. If A is a ∗-element of M then every minimal prime over A is a minimal prime.

Proof. Let P be a minimal prime element of M over A. We know by Theorem 8 and Theorem 18, a ∗-element A is a Baer element and (A : I_M) is a radical element. Let x ∈ L, be such that xI_M ⊆ P. But P is a minimal prime over A. Then by Theorem 2 there exists y ∈ L, such that yI_M ∉ P and x^n yI_M ⊆ A i.e. x^n y ⊆ A : I_M. So x^n y^n ⊆ A : I_M i.e. xy ⊆ √((A : I_M)) = (A : I_M). By hypothesis, xy is compact and xyI_M ⊆ A = 0_F, for some filter F of L, such that 0 ∉ F. Hence xyI_M d = 0_M for some d ∈ F. We show that there is no compact element x in F such that xI_M ⊆ P. Suppose there is compact element z in L such that zI_M ⊆ P and z ∈ F. Then by Theorem 3, 0_M : z ≤ 0_F = A ⊆ P. This contradict the fact that P contains precisely one of zI_M and 0_M : z where z ∈ L. Hence there is no compact element x in F such that xI_M ⊆ P. This implies that dI_M ∉ P. As P is prime, dI_M ∉ P and yI_M ∉ P implies ydI_M ∉ P. Thus xydI_M = 0_M ⊆ P and ydI_M ∉ P. Therefore by Theorem 6, P is minimal prime.
Remark 2. By Theorem 7, we infer that every minimal prime element is a \( * \)-element and it is a Baer element. Therefore by Theorem 21, if \( A \) is the meet of all minimal prime elements containing it, \( A \) is a Baer element.

Notation: For a family \( \{A_a\} \) of Baer elements of \( L \) we define,

\[
\forall A_a = \forall (x I_M, x \in L_+ | 0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M \leq 0_M : x I_M),
\]

for some compact elements \( x I_M \leq A_{a_j} \) and some \( j = 1, 2, \ldots, n \).

The important property of a family of Baer elements is established in the next theorem.

Theorem 22. If \( \{A_a\} \) is a family of Baer elements of \( L \), \( \forall A_a \) is the smallest Baer element greater than each \( A_a \).

Proof. We first show that \( \forall A_a \) is a Baer element greater than each \( A_a \). Let \( x \) be a compact element of \( L \) such that \( x I_M \leq \forall A_a \). Then there exist compact elements \( x_1, x_2, \ldots, x_n \) such that \( 0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M \leq 0_M : x I_M \) and \( x I_M \leq A_{a_j} \) \( j = 1, 2, \ldots, n \). Next we show that \( 0_M : (0_M : x I_M) \leq \forall A_a \). Let \( z \) be a compact element in \( L \) such that \( z I_M \leq 0_M : (0_M : x I_M) \). Then \( 0_M : z I_M \leq 0_M : [0_M : (0_M : x I_M)] \). That is \( 0_M : x I_M \leq 0_M : z I_M \) (by Theorem 1, \( x \) and \( (\forall x) \)). Therefore \( 0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M \leq 0_M : z I_M \). This implies that \( z I_M \leq \forall A_a \). Thus \( 0_M : (0_M : x I_M) \leq \forall A_a \). This shows that \( \forall A_a \) is a Baer element. Let \( z \) be a compact element in \( L \) such that \( z I_M \leq A_a \) for some \( \alpha \). But \( 0_M : z I_M \leq 0_M : z I_M \). Thus \( z I_M \leq \forall A_a \). Hence each \( A_a \leq \forall A_a \). Let \( B \) be a Baer element such that \( A_a \leq B \) for each \( \alpha \) and let \( x \) be a compact element in \( L \) such that \( 0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M \leq 0_M : x I_M \) for some compact elements \( x I_M \leq A_{a_j} \), \( j = 1, 2, \ldots, n \) so that \( x I_M \leq \forall A_a \). Note that \( B \) is a Baer element and the compact element \( (x_1 \lor x_2 \ldots \lor x_n) I_M \leq B \). Hence \( 0_M : (0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M) \leq B \). Again note that \( 0_M : (0_M : x I_M) \leq 0_M : [0_M : (x_1 \lor x_2 \ldots \lor x_n)] I_M \) and \( x I_M \leq 0_M : (0_M : x I_M) \). Therefore \( x I_M \leq B \) and hence \( \forall A_a \leq B \). Consequently \( \forall A_a \) is the smallest Baer element greater than each \( A_a \).

\[\square\]

Theorem 23. For any proper element \( A \in M \), \( \forall \{0_M : (0_M : x I_M) | x \in L_+ \text{ and } x I_M \leq A\} \) is the smallest Baer element greater than \( A \).

Proof. First we show that \( 0_M : (0_M : x I_M) \) is a Baer element i.e. we show that for any \( x \in L_+ \), \( x I_M \leq 0_M : (0_M : x I_M) \) implies \( 0_M : (0_M : x I_M) \leq 0_M : (0_M : x I_M) \) which holds obviously. Hence by Theorem 22, \( B = \forall \{0_M : (0_M : x I_M) | x \in L_+ \text{ and } x I_M \leq A\} \) is the smallest Baer element containing each \( 0_M : (0_M : x I_M) \) for \( x I_M \leq A \). Let a compact element \( x \) in \( L \) be such that \( x I_M \leq A \). Then we have \( x I_M \leq 0_M : (0_M : x I_M) \leq B \). Thus \( A \leq B \). Let \( z I_M \) be a Baer element in \( M \) such that \( A \leq z I_M \) and let \( y \) be a compact element in \( L \) such that \( y I_M \leq B \). Then \( 0_M : (z_1 \lor z_2 \ldots \lor z_n) I_M \leq 0_M : y I_M \), for some compact elements \( z_I_M \leq 0_M : (0_M : x I_M) \), where \( i = 1, 2, \ldots, n \). Thus \( 0_M : x I_M \leq 0_M : z_I_M \) for each \( i \). This gives

\[
0_M : (x_1 \lor x_2 \ldots \lor x_n) I_M \leq 0_M : x_1 I_M \land 0_M : x_2 I_M \land \ldots 0_M : x_n I_M
\]

\[
\leq 0_M : z_1 I_M \land 0_M : z_2 I_M \land \ldots \land 0_M : z_n I_M
\]

\[
= 0_M : (z_1 \lor z_2 \ldots \lor z_n) I_M \leq 0_M : y I_M.
\]
Thus if \( x = x_1 \vee x_2 \vee \ldots \vee x_n \) is compact element such that \( xI_M = (x_1 \vee x_2 \vee \ldots \vee x_n)I_M \leq A \leq zI_M \), we get \( 0_M : xI_M \leq 0_M : yI_M \). As \( zI_M \) is a Baer element we have

\[
yI_M \leq 0_M : (0_M : yI_M) \leq 0_M : (0_M : xI_M) \leq zI_M.
\]

Therefore \( B \leq zI_M \). This shows that \( \forall \{0_M : (0_M : xI_M) \mid x \in L_+ \) and \( xI_M \leq A \} \) is the smallest Baer element greater than \( A \)

Notation: For a family \( \{A_a\} \) of closed elements of \( M \) we define,

\[
A \triangledown B = \bigvee \{zI_M, z \in L_+ \mid 0_M : (x \vee y)I_M \leq 0_M : zI_M \}
\]

for some \( xI_M \leq A \) and \( yI_M \leq B \). Then we have the following important result.

The property of closed elements is proved in the next theorem.

**Theorem 24.** If \( A \) and \( B \) are closed elements of \( M \) \( A \triangledown B \) is the smallest closed element greater than \( A \) as well as \( B \).

**Proof.** We show that \( A \triangledown B \) is greater closed element than \( A \) as well as \( B \). Let \( C = A \triangledown B \). We always have \( C \leq 0_M : (0_M : C) \) where \( C \in M \). Let \( x \) be compact element in \( L \) such that \( xI_M \leq 0_M : (0_M : C) \). Then \( 0_M : C \leq 0_M : xI_M \). This implies that

\[
0_M : (y \vee z)I_M \leq 0_M : C \leq 0_M : xI_M
\]

where \( y, z \in L_+ \), \( yI_M \leq A \) and \( zI_M \leq B \). But \( yI_M \leq A \triangledown B \), \( zI_M \leq A \triangledown B \). Hence

\[
0_M : (r \vee s)I_M \leq 0_M : yI_M \text{ and } 0_M : (u \vee v)I_M \leq 0_M : zI_M
\]

where \( rI_M, uI_M \leq A \) and \( sI_M, vI_M \leq B \). Therefore \( 0_M : (r \vee s)I_M \wedge 0_M : (u \vee v)I_M \leq 0_M : yI_M \wedge 0_M : zI_M \). Consequently

\[
0_M : (r \vee s \vee u \vee v)I_M \leq 0_M : (y \vee z)I_M \leq 0_M : xI_M,
\]

where \( (r \vee u)I_M \leq A \) and \( (s \vee v)I_M \leq B \). This implies that \( xI_M \leq C \). Hence \( 0_M : (0_M : C) \leq C \).

This gives \( 0_M : (0_M : C) = C \) and \( C \) is closed. As \( 0_M : sI_M \leq 0_M : sI_M \) for any element \( s \) in \( L \), it follows that \( A, B \leq A \triangledown B \). Suppose that \( W \) is closed element such that \( A, B \leq W \) and let \( x \in L_+ \) be such that \( 0_M : (u \vee v)I_M \leq 0_M : xI_M \) for some \( uI_M \leq A \) and \( vI_M \leq B \). Note that \( W \) is a closed element and \( (u \vee v)I_M \leq W \). Hence we have \( 0_M : [0_M : (u \vee v)I_M] \leq 0_M : (0_M : W) = W \). Again note that \( 0_M : (0_M : xI_M) \leq 0_M : [0_M : (u \vee v)I_M] \leq W \) and \( xI_M \leq 0_M : (0_M : xI_M) \). Therefore \( xI_M \leq W \) and hence \( A \triangledown B \leq W \). Consequently, it proves that \( A \triangledown B \) is the smallest closed element greater than \( A \) as well as \( B \).

**Theorem 25.** If \( A \) and \( B \) are closed elements of \( M \) then \( A \triangledown B = 0_M : [0_M : (A \lor B)] \).

**Proof.** By Theorem 24, we have \( A \lor B \leq A \triangledown B \). Hence \( 0_M : [0_M : (A \lor B)] \leq A \triangledown B \). As \( A \triangledown B \) is a closed element. Let \( xI_M \leq A \triangledown B, x \in L_+ \). Then \( 0_M : (u \lor v)I_M \leq 0_M : xI_M \), for some \( uI_M \leq A \) and \( vI_M \leq B \). Consequently, we have

\[
xI_M \leq 0_M : (0_M : xI_M) \leq 0_M : [0_M : (u \lor v)I_M] \leq 0_M : [0_M : (A \lor B)].
\]

Hence \( A \triangledown B \leq (0_M : [0_M : (A \lor B)]) \). Thus \( A \triangledown B = 0_M : [0_M : (A \lor B)] \).
ACKNOWLEDGEMENTS The authors thank the readers of European Journal of Pure and Applied Mathematics, for making our journal successful. We dedicate this research article to Prof Dr U Tekir & Prof Dr C Jayram.

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