Classical 2-Absorbing Submodules of Modules over Commutative Rings

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Abstract. In this article, all rings are commutative with nonzero identity. Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is called a classical prime submodule, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. We introduce the concept of “classical 2-absorbing submodules” as a generalization of "classical prime submodules". We say that a proper submodule $N$ of $M$ is a classical 2-absorbing submodule if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$.

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1. Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let $R$ be a commutative ring and $M$ be an $R$-module. A proper submodule $N$ of $M$ is said to be a prime submodule, if for each element $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{ r \in R \mid rM \subseteq N \}$. A proper submodule $N$ of $M$ is called a classical prime submodule, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [9, 10] (see also, [11], in which, the notion of “weakly prime submodules” is investigated). For more information on weakly prime submodules, the reader is referred to [3, 4, 12].

Badawi gave a generalization of prime ideals in [5] and said such ideals 2-absorbing ideals. A proper ideal $I$ of $R$ is a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. He proved that $I$ is a 2-absorbing ideal of $R$ if and only if

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whenever $I_1, I_2, I_3$ are ideals of $R$ with $I_1I_2I_3 \subseteq I$, then $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. Anderson and Badawi [2] generalized the notion of 2-absorbing ideals to $n$-absorbing ideals. A proper ideal $I$ of $R$ is called an $n$-absorbing (resp. a strongly $n$-absorbing) ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of $R$), then there are $n$ of the $x_i$’s (resp. $n$ of the $I_i$’s) whose product is in $I$. The reader is referred to [6–8] for more concepts related to $2$-absorbing ideals. Yousefian Darani and Soheilnia in [13] extended 2-absorbing ideals to 2-absorbing submodules. A proper submodule $N$ of $M$ is called a 2-absorbing submodule of $M$ if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$. Generally, a proper submodule $N$ of $M$ is called an $n$-absorbing submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N :_R M)$ or there are $n-1$ of the $a_i$’s whose product with $m$ is in $N$, see [14]. Several authors investigated properties of 2-absorbing submodules, for example [15].

In this paper we introduce the definition of classical 2-absorbing submodules. A proper submodule $N$ of an $R$-module $M$ is called classical 2-absorbing submodule if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$. Clearly, every classical prime submodule is a classical 2-absorbing submodule. We show that every Noetherian $R$-module $M$ contains a finite number of minimal classical 2-absorbing submodules (Theorem 3). Further, we give the relationship between classical 2-absorbing submodules, classical prime submodules and 2-absorbing submodules (Proposition 2, Proposition 7). Moreover, we characterize classical 2-absorbing submodules in (Theorem 2, Theorem 4). In (Theorem 7, Theorem 8) we investigate classical 2-absorbing submodules of a finite direct product of modules.

2. Characterizations of Classical 2-Absorbing Submodules

First of all we give a module which has no classical 2-absorbing submodule.

**Example 1.** Let $p$ be a fixed prime integer and $N_0 = \mathbb{N} \cup \{0\}$. Then

$$E(p) := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^n} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \text{ and } n \in N_0 \right\}$$

is a nonzero submodule of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. For each $t \in N_0$, set

$$G_t := \left\{ \alpha \in \mathbb{Q}/\mathbb{Z} \mid \alpha = \frac{r}{p^t} + \mathbb{Z} \text{ for some } r \in \mathbb{Z} \right\}.$$

Notice that for each $t \in N_0$, $G_t$ is a submodule of $E(p)$ generated by $\frac{1}{p^t} + \mathbb{Z}$ for each $t \in N_0$. Each proper submodule of $E(p)$ is equal to $G_i$ for some $i \in N_0$ (see, [17, Example 7.10]). However, no $G_t$ is a classical 2-absorbing submodule of $E(p)$. Indeed, $\frac{1}{p^{t+1}} + \mathbb{Z} \in E(p)$. Then $p^3 \left( \frac{1}{p^{t+1}} + \mathbb{Z} \right) = \frac{1}{p^t} + \mathbb{Z} \notin G_t$, but $p^2 \left( \frac{1}{p^{t+1}} + \mathbb{Z} \right) = \frac{1}{p^t} + \mathbb{Z} \notin G_t$.

**Theorem 1.** Let $f : M \to M'$ be an epimorphism of $R$-modules.

(i) If $N'$ is a classical 2-absorbing submodule of $M'$, then $f^{-1}(N')$ is a classical 2-absorbing submodule of $M$. 


(i) If $N$ is a classical 2-absorbing submodule of $M$ containing $\text{Ker}(f)$, then $f(N)$ is a classical 2-absorbing submodule of $M'$.

**Proof.** (i) Since $f$ is epimorphism, $f^{-1}(N')$ is a proper submodule of $M$. Let $a, b, c \in R$ and $m \in M$ such that $abcm \in f^{-1}(N')$. Then $abcf(m) \in N'$. Hence $abf(m) \in N' - \text{Ker}(f)$ or $acf(m) \in N'$ or $bcf(m) \in N'$, and thus $abm \in f^{-1}(N')$ or $acm \in f^{-1}(N')$ or $bcm \in f^{-1}(N')$. So, $f^{-1}(N')$ is a classical 2-absorbing submodule of $M$.

(ii) Let $a, b, c \in R$ and $m' \in M'$ be such that $abcm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abcm) \in f(N)$. Since $\text{Ker}(f) \subseteq N$, we have $abcm \in N$. It implies that $abm \in N$ or $acm \in N$ or $bcm \in N$. Hence $abm \in f(N)$. Consequently $f(N)$ is a classical 2-absorbing submodule of $M'$.

As an immediate consequence of Theorem 1 we have the following corollary.

**Corollary 1.** Let $M$ be an $R$-module and $L \subseteq N$ be submodules of $M$. Then $N$ is a classical 2-absorbing submodule of $M$ if and only if $N/L$ is a classical 2-absorbing submodule of $M/L$.

**Proposition 1.** Let $M$ be an $R$-module and $N_1, N_2$ be classical prime submodules of $M$. Then $N_1 \cap N_2$ is a classical 2-absorbing submodule of $M$.

**Proof.** Let for some $a, b, c \in R$ and $m \in M$, $abcm \in N_1 \cap N_2$. Since $N_1$ is a classical prime submodule, then we may assume that $am \in N_1$. Likewise, assume that $bm \in N_2$. Hence $abm \in N_1 \cap N_2$ which implies $N_1 \cap N_2$ is a classical 2-absorbing submodule.

**Proposition 2.** Let $N$ be a proper submodule of an $R$-module $M$.

(i) If $N$ is a 2-absorbing submodule of $M$, then $N$ is a classical 2-absorbing submodule of $M$.

(ii) $N$ is a classical prime submodule of $M$ if and only if $N$ is a 2-absorbing submodule of $M$ and $(N : R M)$ is a prime ideal of $R$.

**Proof.** (i) Assume that $N$ is a 2-absorbing submodule of $M$. Let $a, b, c \in R$ and $m \in M$ such that $abcm \in N$. Therefore either $acm \in N$ or $bcm \in N$ or $ab \in (N : R M)$. The first two cases lead us to the claim. In the third case we have that $abm \in N$. Consequently $N$ is a classical 2-absorbing submodule.

(ii) It is evident that if $N$ is classical prime, then it is 2-absorbing. Also, [3, Lemma 2.1] implies that $(N : R M)$ is a prime ideal of $R$. Assume that $N$ is a 2-absorbing submodule of $M$ and $(N : R M)$ is a prime ideal of $R$. Let $abm \in N$ for some $a, b \in R$ and $m \in M$ such that neither $am \in N$ nor $bm \in N$. Then $ab \in (N : R M)$ and so either $a \in (N : R M)$ or $b \in (N : R M)$. This contradiction shows that $N$ is classical prime.

The following example shows that the converse of Proposition 2(ii) is not true.

**Example 2.** Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \oplus \mathbb{Z}_q \oplus \mathbb{Q}$ where $p, q$ are two distinct prime integers. One can easily see that the zero submodule of $M$ is a classical 2-absorbing submodule. Notice that $pq(1, 1, 0) = (0, 0, 0)$, but $p(1, 1, 0) \neq (0, 0, 0)$, $q(1, 1, 0) \neq (0, 0, 0)$ and $pq(1, 1, 1) \neq 0$. So the zero submodule of $M$ is not 2-absorbing. Also, part (ii) of Proposition 2 shows that the zero submodule is not a classical prime submodule. Hence the two concepts of classical prime submodules and of classical 2-absorbing submodules are different in general.
Let $M$ be an $R$-module and $N$ a submodule of $M$. For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_R a)$. It is easy to see that $(N :_M a)$ is a submodule of $M$ containing $N$.

**Theorem 2.** Let $M$ be an $R$-module and $N$ be a proper submodule of $M$. The following conditions are equivalent:

(i) $N$ is classical 2-absorbing;

(ii) For every $a, b, c \in R$, $(N :_M abc) = (N :_M ab) \cup (N :_M ac) \cup (N :_M bc)$;

(iii) For every $a, b \in R$ and $m \in M$ with $abm \notin N$, $(N :_R abm) = (N :_R am) \cup (N :_R bm)$;

(iv) For every $a, b \in R$ and $m \in M$ with $abm \notin N$, $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R bm)$;

(v) For every $a, b \in R$ and every ideal $I$ of $R$ and $m \in M$ with $abIm \subseteq N$, either $abm \in N$ or $aIm \subseteq N$ or $bIm \subseteq N$;

(vi) For every $a \in R$ and every ideal $I$ of $R$ and $m \in M$ with $aIm \notin N$, $(N :_R aIm) = (N :_R am)$ or $(N :_R aIm) = (N :_R Im)$;

(vii) For every $a \in R$ and every ideals $I, J$ of $R$ and $m \in M$ with $aIJm \subseteq N$, either $aIm \subseteq N$ or $aJm \subseteq N$ or $IJm \subseteq N$;

(viii) For every ideals $I, J$ and $m \in M$ with $IJm \notin N$, $(N :_R IJm) = (N :_R Im)$ or $(N :_R IJm) = (N :_R Jm)$;

(ix) For every ideals $I, J, K$ of $R$ and $m \in M$ with $IJKm \subseteq N$, either $IJm \subseteq N$ or $Ikm \subseteq N$ or $JKm \subseteq N$;

(x) For every $m \in M \setminus N$, $(N :_R m)$ is a 2-absorbing ideal of $R$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $N$ is a classical 2-absorbing submodule of $M$. Let $m \in (N :_M abc)$. Then $abcm \in N$. Hence $abm \in N$ or $acm \in N$ or $bcm \in N$. Therefore $m \in (N :_M ab)$ or $m \in (N :_M ac)$ or $m \in (N :_M bc)$. Consequently,

$$(N :_M abc) = (N :_M ab) \cup (N :_M ac) \cup (N :_M bc).$$

(ii) $\Rightarrow$ (iii) Let $abm \notin N$ for some $a, b \in R$ and $m \in M$. Assume that $x \in (N :_R abm)$. Then $abxm \in N$, and so $m \in (N :_M abx)$. Since $abm \notin N$, $m \notin (N :_M ab)$. Thus by part (i), $m \in (N :_M ax)$ or $m \in (N :_M bx)$, whence $x \in (N :_R am)$ or $x \in (N :_R bm)$. Therefore $(N :_R abm) = (N :_R am) \cup (N :_R bm)$.

(iii) $\Rightarrow$ (iv) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(iv) $\Rightarrow$ (v) Let for some $a, b \in R$, an ideal $I$ of $R$ and $m \in M$, $abIm \subseteq N$. Hence $I \subseteq (N :_R abm)$. If $abm \in N$, then we are done. Assume that $abm \notin N$. Therefore by part (iv) we have that $I \subseteq (N :_R am)$ or $I \subseteq (N :_R bm)$, i.e., $alm \subseteq N$ or $bIm \subseteq N$.

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ix) Have proofs similar to that of the previous implications.
Corollary 2. Let $R$ be a ring and $I$ be a proper ideal of $R$.

(i) $RI$ is a classical 2-absorbing submodule of $R$ if and only if $I$ is a 2-absorbing ideal of $R$.

(ii) Every proper ideal of $R$ is 2-absorbing if and only if for every $R$-module $M$ and every proper submodule $N$ of $M$, $N$ is a classical 2-absorbing submodule of $M$.

Proof. (i) Let $I$ be a classical 2-absorbing submodule of $R$. Then by Theorem 2, $(I :_R 1) = I$ is a 2-absorbing ideal of $R$. For the converse see part (i) of Proposition 2.

(ii) Assume that every proper ideal of $R$ is 2-absorbing. Let $N$ be a proper submodule of an $R$-module $M$. Since for every $m \in M \setminus N$, $(N :_R m)$ is a proper ideal of $R$, then it is a 2-absorbing ideal of $R$. Hence by Theorem 2, $N$ is a classical 2-absorbing submodule of $M$. We have the converse immediately by part (i).

Proposition 3. Let $M$ be an $R$-module and $\{ K_i \mid i \in I \}$ be a chain of classical 2-absorbing submodules of $M$. Then $\cap_{i \in I} K_i$ is a classical 2-absorbing submodule of $M$.

Proof. Suppose that $abcm \in \cap_{i \in I} K_i$ for some $a, b, c \in R$ and $m \in M$. Assume that $abm \notin \cap_{i \in I} K_i$ and $acm \notin \cap_{i \in I} K_i$. Then there are $t, l \in I$ where $abm \notin K_l$ and $acm \notin K_l$. Hence, for every $K_i \subseteq K_l$ and every $K_d \subseteq K_l$ we have that $abm \notin K_d$ and $acm \notin K_d$. Thus, for every submodule $K_h$ such that $K_h \subseteq K_l$ and $K_h \subseteq K_i$ we get $bcm \notin K_h$. Hence $bcm \in \cap_{i \in I} K_i$.

A classical 2-absorbing submodule of $M$ is called minimal, if for any classical 2-absorbing submodule $K$ of $M$ such that $K \subseteq N$, then $K = N$. Let $L$ be a classical 2-absorbing submodule of $M$. Set

$$\Gamma = \{ K \mid K \text{ is a classical 2-absorbing submodule of } M \text{ and } K \subseteq L \}.$$ 

If $\{ K_i : i \in I \}$ is any chain in $\Gamma$, then $\cap_{i \in I} K_i$ is in $\Gamma$, by Proposition 3. By Zorn’s Lemma, $\Gamma$ contains a minimal member which is clearly a minimal classical 2-absorbing submodule of $M$. Thus, every classical 2-absorbing submodule of $M$ contains a minimal classical 2-absorbing submodule of $M$. If $M$ is a finitely generated, then it is clear that $M$ contains a minimal classical 2-absorbing submodule.

Theorem 3. Let $M$ be a Noetherian $R$-module. Then $M$ contains a finite number of minimal classical 2-absorbing submodules.

Proof. Suppose that the result is false. Let $\Gamma$ denote the collection of proper submodules $N$ of $M$ such that the module $M/N$ has an infinite number of minimal classical 2-absorbing submodules. Since $0 \in \Gamma$ we get $\Gamma \neq \emptyset$. Therefore $\Gamma$ has a maximal member $T$, since $M$ is a Noetherian $R$-module. It is clear that $T$ is not a classical 2-absorbing submodule. Therefore, there exists an element $m \in M \setminus T$ and ideals $I, J, K$ in $R$ such that $IKm \subseteq T$ but $IJm \not\subseteq T$, $IKm \not\subseteq T$ and $JKm \not\subseteq T$. The maximality of $T$ implies that $M/(T + IJm), M/(T + IKm)$
and $M/(T+JKm)$ have only finitely many minimal classical 2-absorbing submodules. Suppose $P/T$ be a minimal classical 2-absorbing submodule of $M/T$. So $IJKm \subseteq T \subseteq P$, which implies that $IJm \subseteq P$ or $IKm \subseteq P$ or $JKm \subseteq P$. Thus $P/(T+IJm)$ is a minimal classical 2-absorbing submodule of $M/(T+IJm)$ or $P/(T+IKm)$ is a minimal classical 2-absorbing submodule of $M/(T+IKm)$ or $P/(T+JKm)$ is a minimal classical 2-absorbing submodule of $M/(T+JKm)$. Thus, there are only a finite number of possibilities for the submodule $P$. This is a contradiction. 

We recall from [5] that if $I$ is a 2-absorbing ideal of a ring $R$, then either $\sqrt{I} = P$ where $P$ is a prime ideal of $R$ or $\sqrt{I} = P_1 \cap P_2$ where $P_1$, $P_2$ are the only distinct minimal prime ideals of $I$.

**Corollary 3.** Let $N$ be a classical 2-absorbing submodule of an $R$-module $M$. Suppose that $m \in M \setminus N$ and $\sqrt{(N:_R m)} = P$ where $P$ is a prime ideal of $R$ and $(N:_R m) \neq P$. Then for each $x \in \sqrt{(N:_R m)} \setminus (N:_R m)$, $(N:_R xm)$ is a prime ideal of $R$ containing $P$. Furthermore, either $(N:_R xm) \subseteq (N:_R ym)$ or $(N:_R ym) \subseteq (N:_R xm)$ for every $x, y \in \sqrt{(N:_R m)} \setminus (N:_R m)$.

**Proof.** By Theorem 2 and [5, Theorem 2.5].

**Corollary 4.** Let $N$ be a classical 2-absorbing submodule of an $R$-module $M$. Suppose that $m \in M \setminus N$ and $\sqrt{(N:_R m)} = P_1 \cap P_2$ where $P_1$ and $P_2$ are the only nonzero distinct prime ideals of $R$ that are minimal over $(N:_R m)$ in part (ii). Then for each $x \in \sqrt{(N:_R m)} \setminus (N:_R m)$, $(N:_R xm)$ is a prime ideal of $R$ containing $P_1$ and $P_2$. Furthermore, either $(N:_R xm) \subseteq (N:_R ym)$ or $(N:_R ym) \subseteq (N:_R xm)$ for every $x, y \in \sqrt{(N:_R m)} \setminus (N:_R m)$.

**Proof.** By Theorem 2 and [5, Theorem 2.6].

An $R$-module $M$ is called a **multiplication module** if every submodule $N$ of $M$ has the form $IM$ for some ideal $I$ of $R$. Let $N$ and $K$ be submodules of a multiplication $R$-module $M$ with $N = I_1M$ and $K = I_2M$ for some ideals $I_1$ and $I_2$ of $R$. The product of $N$ and $K$ denoted by $NK$ is defined by $NK = I_1I_2M$. Then by [1, Theorem 3.4], the product of $N$ and $K$ is independent of presentations of $N$ and $K$.

**Proposition 4.** Let $M$ be a multiplication $R$-module and $N$ be a proper submodule of $M$. The following conditions are equivalent:

(i) $N$ is a classical 2-absorbing submodule of $M$;

(ii) If $N_1N_2N_3m \subseteq N$ for some submodules $N_1$, $N_2$, $N_3$ of $M$ and $m \in M$, then either $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$.

**Proof.** (i) $\Rightarrow$ (ii) Let $N_1N_2N_3m \subseteq N$ for some submodules $N_1$, $N_2$, $N_3$ of $M$ and $m \in M$. Since $M$ is multiplication, there are ideals $I_1$, $I_2$, $I_3$ of $R$ such that $N_1 = I_1M$, $N_2 = I_2M$ and $N_3 = I_3M$. Therefore $I_1I_2I_3m \subseteq N$, and so either $I_1I_2m \subseteq N$ or $I_1I_3m \subseteq N$ or $I_2I_3m \subseteq N$. Hence $N_1N_2m \subseteq N$ or $N_1N_3m \subseteq N$ or $N_2N_3m \subseteq N$.

(ii) $\Rightarrow$ (i) Suppose that $I_1I_2I_3m \subseteq N$ for some ideals $I_1$, $I_2$, $I_3$ of $R$ and some $m \in M$. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 := I_3M$ in part (ii).
In [16], Quartararo et al. said that a commutative ring $R$ is a $u$-ring provided $R$ has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a um-ring is a ring $R$ with the property that an $R$-module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a $u$-ring. Moreover, they proved that every Prüfer domain is a $u$-domain. Also, any ring which contains an infinite field as a subring is a $u$-ring, [17, Exercise 3.63].

**Theorem 4.** Let $R$ be a um-ring, $M$ be an $R$-module and $N$ be a proper submodule of $M$. The following conditions are equivalent:

(i) $N$ is classical 2-absorbing;

(ii) For every $a, b, c \in R$, $(N:_{M} abc) = (N:_{M} ab) \lor (N:_{M} abc) = (N:_{M} ac)$ or $(N:_{M} abc) = (N:_{M} bc)$;

(iii) For every $a, b, c \in R$ and every submodule $L$ of $M$, $abcL \subseteq N$ implies that $abL \subseteq N$ or $acL \subseteq N$ or $bcL \subseteq N$;

(iv) For every $a, b \in R$ and every submodule $L$ of $M$ with $abL \not\subseteq N$, $(N:_{R} abL) = (N:_{R} aL)$ or $(N:_{R} abL) = (N:_{R} bL)$;

(v) For every $a, b \in R$, every ideal $I$ of $R$ and every submodule $L$ of $M$, $abIL \subseteq N$ implies that $abL \subseteq N$ or $aIL \subseteq N$ or $bIL \subseteq N$;

(vi) For every $a \in R$, every ideal $I$ of $R$ and every submodule $L$ of $M$ with $aIL \not\subseteq N$, $(N:_{R} aIL) = (N:_{R} aL)$ or $(N:_{R} aIL) = (N:_{R} IL)$;

(vii) For every $a \in R$, every ideals $I, J$ of $R$ and every submodule $L$ of $M$, $aIJL \subseteq N$ implies that $aIL \subseteq N$ or $aIL \subseteq N$ or $IJL \subseteq N$;

(viii) For every ideals $I, J$ of $R$ and every submodule $L$ of $M$ with $IJL \not\subseteq N$, $(N:_{R} IJL) = (N:_{R} IL)$ or $(N:_{R} IJL) = (N:_{R} JL)$;

(ix) For every ideals $I, J, K$ of $R$ and every submodule $L$ of $M$, $IJKL \subseteq N$ implies that $IJL \subseteq N$ or $IKL \subseteq N$ or $JKL \subseteq N$;

(x) For every submodule $L$ of $M$ not contained in $N$, $(N:_{R} L)$ is a 2-absorbing ideal of $R$.

**Proof.** Similar to the proof of Theorem 2.

**Proposition 5.** Let $R$ be a um-ring and $N$ be a proper submodule of an $R$-module $M$. Then $N$ is a classical 2-absorbing submodule of $M$ if and only if $N$ is a 3-absorbing submodule of $M$ and $(N:_{R} M)$ is a 2-absorbing ideal of $R$.

**Proof.** It is trivial that if $N$ is classical 2-absorbing, then it is 3-absorbing. Also, Theorem 4 implies that $(N:_{R} M)$ is a 2-absorbing ideal of $R$. Now, assume that $N$ is a 3-absorbing submodule of $M$ and $(N:_{R} M)$ is a 2-absorbing ideal of $R$. Let $a_{1}a_{2}a_{3}m \in N$ for some $a_{1}, a_{2}, a_{3} \in R$ and $m \in M$ such that neither $a_{1}a_{2}m \in N$ nor $a_{1}a_{3}m \in N$ nor $a_{2}a_{3}m \in N$. Then $a_{1}a_{2}a_{3} \in (N:_{R} M)$
and so either \( a_1a_2 \in (N :_R M) \) or \( a_1a_3 \in (N :_R M) \) or \( a_2a_3 \in (N :_R M) \). This contradiction shows that \( N \) is classical 2-absorbing.

\[ \square \]

**Proposition 6.** Let \( M \) be an \( R \)-module and \( N \) be a classical 2-absorbing submodule of \( M \). The following conditions hold:

(i) For every \( a, b, c \in R \) and \( m \in M \), \((N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)\);

(ii) If \( R \) is a u-ring, then for every \( a, b, c \in R \) and \( m \in M \), \((N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)\).

Proof. (i) Let \( a, b, c \in R \) and \( m \in M \). Suppose that \( r \in (N :_R abcm) \). Then \( abc(rm) \in N \).

So, either \( ab(rm) \in N \) or \( ac(rm) \in N \) or \( bc(rm) \in N \). Therefore, either \( r \in (N :_R abm) \) or \( r \in (N :_R acm) \) or \( r \in (N :_R bcm) \). Consequently \((N :_R abcm) = (N :_R abm) \cup (N :_R acm) \cup (N :_R bcm)\).

(ii) Use part (i).

\[ \square \]

**Proposition 7.** Let \( R \) be a u-ring, \( M \) be a multiplication \( R \)-module and \( N \) be a proper submodule of \( M \). The following conditions are equivalent:

(i) \( N \) is a classical 2-absorbing submodule of \( M \);

(ii) If \( N_1N_2N_3N_4 \subseteq N \) for some submodules \( N_1, N_2, N_3, N_4 \) of \( M \), then either \( N_1N_2N_4 \subseteq N \) or \( N_1N_3N_4 \subseteq N \) or \( N_2N_3N_4 \subseteq N \);

(iii) If \( N_1N_2N_3 \subseteq N \) for some submodules \( N_1, N_2, N_3 \) of \( M \), then either \( N_1N_2 \subseteq N \) or \( N_1N_3 \subseteq N \) or \( N_2N_3 \subseteq N \);

(iv) \( N \) is a 2-absorbing submodule of \( M \);

(v) \((N :_R M)\) is a 2-absorbing ideal of \( R \).

Proof. (i) \( \Rightarrow \) (ii) Let \( N_1N_2N_3N_4 \subseteq N \) for some submodules \( N_1, N_2, N_3, N_4 \) of \( M \). Since \( M \) is multiplication, there are ideals \( I_1, I_2, I_3 \) of \( R \) such that \( N_1 = I_1M, N_2 = I_2M \) and \( N_3 = I_3M \). Therefore \( I_1I_2I_3 \subseteq N \) and so \( I_1I_2N_4 \subseteq N \) or \( I_1I_3N_4 \subseteq N \) or \( I_2I_3N_4 \subseteq N \). Thus by Theorem 4, either \( N_1N_2N_4 \subseteq N \) or \( N_1N_3N_4 \subseteq N \) or \( N_2N_3N_4 \subseteq N \).

(ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) Suppose that \( I_1I_2K \subseteq N \) for some ideals \( I_1, I_2 \) of \( R \) and some submodule \( K \) of \( M \). It is sufficient to set \( N_1 := I_1M, N_2 := I_2M \) and \( N_3 = K \) in part (iii).

(iv) \( \Rightarrow \) (i) By part (i) of Proposition 2.

(v) \( \Rightarrow \) (iv) Let \( I_1I_2K \subseteq N \) for some ideals \( I_1, I_2 \) of \( R \) and some submodule \( K \) of \( M \). Since \( M \) is multiplication, then there is an ideal \( I_3 \) of \( R \) such that \( K = I_3M \). Hence \( I_1I_2I_3 \subseteq (N :_R M) \) which implies that either \( I_1I_2 \subseteq (N :_R M) \) or \( I_1I_3 \subseteq (N :_R M) \) or \( I_2I_3 \subseteq (N :_R M) \). If \( I_1I_2 \subseteq (N :_R M) \), then we are done. So, suppose that \( I_1I_3 \subseteq (N :_R M) \). Thus \( I_1I_3M = I_1K \subseteq N \). Similarly if \( I_2I_3 \subseteq (N :_R M) \), then we have \( I_2K \subseteq N \).

\[ \square \]
Definition 1. Let $R$ be a um-ring, $M$ be an $R$-module and $S$ be a subset of $M \setminus \{0\}$. If for all ideals $I, J, Q$ of $R$ and all submodules $K, L$ of $M$, $(K + IJL) \cap S \neq \emptyset$ and $(K + IQL) \cap S \neq \emptyset$ and $(K + JQL) \cap S \neq \emptyset$ implies $(K + IJQL) \cap S \neq \emptyset$, then the subset $S$ is called classical 2-absorbing $m$-closed.

Proposition 8. Let $R$ be a um-ring, $M$ be $R$-module and $N$ a submodule of $M$. Then $N$ is a classical 2-absorbing submodule if and only if $M \setminus N$ is a classical 2-absorbing $m$-closed.

Proof. Suppose that $N$ is a classical 2-absorbing submodule of $M$ and $I, J, Q$ are ideals of $R$ and $K, L$ are submodules of $M$ such that $(K + IJL) \cap S \neq \emptyset$ and $(K + IQL) \cap S \neq \emptyset$ and $(K + JQL) \cap S \neq \emptyset$ where $S = M \setminus N$. Assume that $(K + IJQL) \cap S = \emptyset$. Then $K + IJQL \subseteq N$ and so $K \subseteq N$ and $IJQL \subseteq N$. Since $N$ is a classical 2-absorbing submodule, we get $IJL \subseteq N$ or $IQL \subseteq N$ or $JQL \subseteq N$. If $IJL \subseteq N$, then we get $(K + IJL) \cap S = \emptyset$, since $K \subseteq N$. This is a contradiction. By the other cases we get similar contradictions. Now for the converse suppose that $S = M \setminus N$ is a classical 2-absorbing $m$-closed and assume that $IJQL \subseteq N$ for some ideals $I, J, Q$ of $R$ and submodule $L$ of $M$. Then we get for submodule $K = (0)$, $K + IJQL \subseteq N$. Thus $(K + IJQL) \cap S = \emptyset$. Since $S$ is a classical 2-absorbing $m$-closed, $(K + IJL) \cap S = \emptyset$ or $(K + IQL) \cap S = \emptyset$ or $(K + JQL) \cap S = \emptyset$. Hence $IJL \subseteq N$ or $IQL \subseteq N$ or $JQL \subseteq N$. So $N$ is a classical 2-absorbing submodule.

Proposition 9. Let $R$ be a um-ring, $M$ be an $R$-module, $N$ a submodule of $M$ and $S = M \setminus N$. The following conditions are equivalent:

(i) $N$ is a classical 2-absorbing submodule of $M$;

(ii) $S$ is a classical 2-absorbing $m$-closed;

(iii) For every ideals $I, J, Q$ of $R$ and every submodule $L$ of $M$, if $IJL \cap S \neq \emptyset$ and $IQL \cap S \neq \emptyset$ and $JQL \cap S \neq \emptyset$, then $IJQL \cap S \neq \emptyset$;

(iv) For every ideals $I, J, Q$ of $R$ and every $m \in M$, if $IJm \cap S \neq \emptyset$ and $IQm \cap S \neq \emptyset$ and $JQm \cap S \neq \emptyset$, then $IJQm \cap S \neq \emptyset$.

Proof. It follows from the previous Proposition, Theorem 2 and Theorem 4.

Theorem 5. Let $R$ be a um-ring, $M$ be an $R$-module and $S$ be a classical 2-absorbing $m$-closed. Then the set of all submodules of $M$ which are disjoint from $S$ has at least one maximal element. Any such maximal element is a classical 2-absorbing submodule.

Proof. Let $\Psi = \{N \mid N$ is a submodule of $M$ and $N \cap S = \emptyset\}$. Then $(0) \in \Psi \neq \emptyset$. Since $\Psi$ is partially ordered by using Zorn’s Lemma we get at least a maximal element of $\Psi$, say $P$, with property $P \cap S = \emptyset$. Now we will show that $P$ is classical 2-absorbing. Suppose that $IJQL \subseteq P$ for ideals $I, J, Q$ of $R$ and submodule $L$ of $M$. Assume that $IJL \nsubseteq P$ or $IQL \nsubseteq P$ or $JQL \nsubseteq P$. Then by the maximality of $P$ we get $(IJL + P) \cap S \neq \emptyset$ and $(IJL + P) \cap S \neq \emptyset$ and $(IJL + P) \cap S \neq \emptyset$. Since $S$ is a classical 2-absorbing $m$-closed we have $(IJL + P) \cap S \neq \emptyset$. Hence $P \cap S \neq \emptyset$, which is a contradiction. Thus $P$ is a classical 2-absorbing submodule of $M$. 

**Theorem 6.** Let $R$ be a um-ring and $M$ be an $R$-module.

(i) If $F$ is a flat $R$-module and $N$ is a classical 2-absorbing submodule of $M$ such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.

(ii) Suppose that $F$ is a faithfully flat $R$-module. Then $N$ is a classical 2-absorbing submodule of $M$ if and only if $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.

**Proof.** (i) Let $a, b, c \in R$. Then we get by Theorem 4, $(N :_M abc) = (N :_M ab)$ or $(N :_M abc) = (N :_M ac)$ or $(N :_M abc) = (N :_M bc)$. Assume that $(N :_M abc) = (N :_M ab)$.

Let $F \otimes N :_{F \otimes M} abc = F \otimes (N :_M ab) = F \otimes (N :_M ab) = (F \otimes N :_{F \otimes M} ab)$.

Again Theorem 4 implies that $F \otimes N$ is a classical 2-absorbing submodule of $F \otimes M$.

(ii) Let $N$ be a classical 2-absorbing submodule of $M$ and assume that $F \otimes N = F \otimes M$. Then $0 \to F \otimes N \xrightarrow{\phi} F \otimes M \to 0$ is an exact sequence. Since $F$ is a faithfully flat module, $0 \to N \xrightarrow{\phi} M \to 0$ is an exact sequence. So $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a classical 2-absorbing submodule by (1). Now for conversely, let $F \otimes N$ be a classical 2-absorbing submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b, c \in R$. Then $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ab)$ or $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ac)$ or $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} bc)$ by Theorem 4. Assume that $(F \otimes N :_{F \otimes M} abc) = (F \otimes N :_{F \otimes M} ab)$.

Hence

$$F \otimes (N :_M ab) = (F \otimes N :_{F \otimes M} ab) = (F \otimes N :_{F \otimes M} ab) = F \otimes (N :_M ab).$$

So $0 \to F \otimes (N :_M ab) \xrightarrow{\phi} F \otimes (N :_M ab) \to 0$ is an exact sequence. Since $F$ is a faithfully flat module, $0 \to (N :_M ab) \xrightarrow{\phi} (N :_M ab) \to 0$ is an exact sequence which implies that $(N :_M ab) = (N :_M ab)$. Consequently $N$ is a classical 2-absorbing submodule of $M$ by Theorem 4.

**Corollary 5.** Let $R$ be a um-ring, $M$ be an $R$-module and $X$ be an indeterminate. If $N$ is a classical 2-absorbing submodule of $M$, then $N[X]$ is a classical 2-absorbing submodule of $M[X]$.

**Proof.** Assume that $N$ is a classical 2-absorbing submodule of $M$. Notice that $R[X]$ is a flat $R$-module. So by Theorem 6, $R[X] \otimes N \simeq N[X]$ is a classical 2-absorbing submodule of $R[X] \otimes M \simeq M[X]$.

For an $R$-module $M$, the set of zero-divisors of $M$ is denoted by $Z_R(M)$.

**Proposition 10.** Let $M$ be an $R$-module, $N$ be a submodule and $S$ be a multiplicative subset of $R$.

(i) If $N$ is a classical 2-absorbing submodule of $M$ such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$. 

[4, Lemma 3.2]
(ii) If $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ such that $Z_R(M/N) \cap S = \emptyset$, then $N$ is a classical 2-absorbing submodule of $M$.

**Proof.** (i) Let $N$ be a classical 2-absorbing submodule of $M$ and $(N :_R M) \cap S = \emptyset$. Suppose that $\frac{a_1}{s_1} \frac{a_2}{s_2} \frac{a_3}{s_3} m \in S^{-1}N$. Then there exist $n \in N$ and $s \in S$ such that $\frac{a_1 a_2 a_3 m}{s_1 s_2 s_3 s_4} = \frac{n}{s'}$. Therefore there exists an $s' \in S$ such that $s'sa_1a_2a_3m = s'ss_2s_3s_4n \in N$. So $a_1a_2a_3(s'm) \in N$ for $s' = s's$. Since $N$ is a classical 2-absorbing submodule we get $a_1a_2(s'm) \in N$ or $a_1a_3(s'm) \in N$ or $a_2a_3(s'm) \in N$. Thus $\frac{a_1a_2m}{s_1s_2s_4} = \frac{a_1a_3(s'm)}{s_1s_2s_3s_4} \in S^{-1}N$ or $\frac{a_2a_3m}{s_2s_3s_4} \in S^{-1}N$.

(ii) Assume that $S^{-1}N$ is a classical 2-absorbing submodule of $S^{-1}M$ and $Z_R(M/N) \cap S = \emptyset$. Let $a, b, c \in R$ and $m \in M$ such that $abc \in N$. Then $\frac{a b c m}{1} \in S^{-1}N$. Therefore $\frac{a b m}{1} \in S^{-1}N$ or $\frac{c \in N}{1} \in S^{-1}N$ or $\frac{b \in N}{1} \in S^{-1}N$. We may assume that $\frac{a b m}{1} \in S^{-1}N$. So there exists $u \in S$ such that $uabm \in N$. But $Z_R(M/N) \cap S = \emptyset$, whence $abm \in N$. Consequently $N$ is a classical 2-absorbing submodule of $M$. \hfill \square

Let $R_i$ be a commutative ring with identity and $M_i$ be an $R_i$-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an $R$-module and each submodule of $M$ is in the form of $N = N_1 \times N_2$ for some submodules $N_1$ of $M_1$ and $N_2$ of $M_2$.

**Theorem 7.** Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an $R$-module where $M_1$ is an $R_1$-module and $M_2$ is an $R_2$-module. Suppose that $N = N_1 \times N_2$ is a proper submodule of $M$. Then the following conditions are equivalent:

(i) $N$ is a classical 2-absorbing submodule of $M$;

(ii) Either $N_1 = M_1$ and $N_2$ is a classical 2-absorbing submodule of $M_2$ or $N_2 = M_2$ and $N_1$ is a classical 2-absorbing submodule of $M_1$ or $N_1$ and $N_2$ are classical prime submodules of $M_1$, $M_2$, respectively.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $N$ is a classical 2-absorbing submodule of $M$ such that $N_2 = M_2$. From our hypothesis, $N$ is proper, so $N_1 \neq M_1$. Set $M' = \frac{M}{(0) \times M_2}$. Hence $N' = \frac{N}{N_1 \times M_2}$ is a classical 2-absorbing submodule of $M'$ by Corollary 1. Also observe that $M' \cong M_1$ and $N' \cong N_1$. Thus $N_1$ is a classical 2-absorbing submodule of $M_1$. Suppose that $N_1 \neq M_1$ and $N_2 \neq M_2$. We show that $N_1$ is a classical prime submodule of $M_1$. Since $N_2 \neq M_2$, there exists $m_2 \in M_2 \setminus N_2$. Let $abm_1 \in N$ for some $a, b \in R_1$ and $m_1 \in M_1$. Thus

$$(a, 1)(b, 1)(1, 0)(m_1, m_2) = (abm_1, 0) \in N = N_1 \times N_2.$$ 

So either $(a, 1)(1, 0)(m_1, m_2) = (am_1, 0) \in N$ or $(b, 1)(1, 0)(m_1, m_2) = (bm_1, 0) \in N$. Hence either $am_1 \in N_1$ or $bm_1 \in N_1$ which shows that $N_1$ is a classical prime submodule of $M_1$. Similarly we can show that $N_2$ is a classical prime submodule of $M_2$.

(ii) $\Rightarrow$ (i) Suppose that $N = N_1 \times M_2$ where $N_1$ is a classical 2-absorbing (resp. classical prime) submodule of $M_1$. Then it is clear that $N$ is a classical 2-absorbing (resp. classical prime) submodule of $M$. Now, assume that $N = N_1 \times N_2$ where $N_1$ and $N_2$ are classical prime submodules of $M_1$ and $M_2$, respectively. Hence $(N_1 \times M_2) \cap (M_1 \times N_2) = N_1 \times N_2 = N$ is a classical 2-absorbing submodule of $M$, by Proposition 1. \hfill \square
Lemma 1. Let \( R = R_1 \times R_2 \times \cdots \times R_n \) be a decomposable ring and \( M = M_1 \times M_2 \times \cdots \times M_n \) be an \( R \)-module where for every \( 1 \leq i \leq n \), \( M_i \) is an \( R_i \)-module, respectively. A proper submodule \( N \) of \( M \) is a classical prime submodule of \( M \) if and only if \( N = \times_{i=1}^n N_i \) such that for some \( k \in \{1, 2, \ldots, n\} \), \( N_k \) is a classical prime submodule of \( M_k \), and \( N_i = M_i \) for every \( i \in \{1, 2, \ldots, n\} \setminus \{k\} \).

Proof. \((\Rightarrow)\) Let \( N \) be a classical prime submodule of \( M \). We know \( N = \times_{i=1}^n N_i \) where for every \( 1 \leq i \leq n \), \( N_i \) is a submodule of \( M_i \), respectively. Assume that \( N_r \) is a proper submodule of \( M_r \) and \( N_s \) is a proper submodule of \( M_s \) for some \( 1 \leq r < s \leq n \). So, there are \( m_r \in M_r \setminus N_r \) and \( m_s \in M_s \setminus N_s \). Since

\[
(0, \ldots, 0, 1_{R_r}, 0, \ldots, 0)(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1_{R_s}, 0, \ldots, 0) = (0, \ldots, 0) \in N,
\]

then either

\[
(0, \ldots, 0, 1_{R_r}, 0, \ldots, 0)(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1_{R_s}, 0, \ldots, 0) = (0, \ldots, 0, m_r, 0, \ldots, 0) \in N
\]

or

\[
(0, \ldots, 0, 1_{R_r}, 0, \ldots, 0)(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0, 1_{R_s}, 0, \ldots, 0) = (0, \ldots, 0, m_s, 0, \ldots, 0) \in N,
\]

which is a contradiction. Hence exactly one of the \( N_i \)'s is proper, say \( N_k \). Now, we show that \( N_k \) is a classical prime submodule of \( M_k \). Let \( abm_k \in N_k \) for some \( a, b \in R_k \) and \( m_k \in M_k \). Therefore

\[
(0, \ldots, 0, a, 0, \ldots, 0)(0, \ldots, 0, b, 0, \ldots, 0)(0, \ldots, 0, m_k, 0, \ldots, 0) = (0, \ldots, 0, abm_k, 0, \ldots, 0) \in N,
\]

and so

\[
(0, \ldots, 0, a, 0, \ldots, 0)(0, \ldots, 0, m_k, 0, \ldots, 0) = (0, \ldots, 0, am_k, 0, \ldots, 0) \in N
\]

or

\[
(0, \ldots, 0, b, 0, \ldots, 0)(0, \ldots, 0, m_k, 0, \ldots, 0) = (0, \ldots, 0, bm_k, 0, \ldots, 0) \in N.
\]

Thus \( am_k \in N_k \) or \( bm_k \in N_k \) which implies that \( N_k \) is a classical prime submodule of \( M_k \).

\((\Leftarrow)\) Is easy. \(\square\)
Theorem 8. Let \( R = R_1 \times R_2 \times \cdots \times R_n \) (\( 2 \leq n < \infty \)) be a decomposable ring and \( M = M_1 \times M_2 \times \cdots \times M_n \) be an \( R \)-module where for every \( 1 \leq i \leq n \), \( M_i \) is an \( R_i \)-module, respectively. For a proper submodule \( N \) of \( M \) the following conditions are equivalent:

(i) \( N \) is a classical 2-absorbing submodule of \( M \);

(ii) Either \( N = \times_{t=1}^{n} N_t \) such that for some \( k \in \{1, 2, \ldots, n\} \), \( N_k \) is a classical 2-absorbing submodule of \( M_k \), and \( N_t = M_t \) for every \( t \in \{1, 2, \ldots, n\} \backslash \{k\} \) or \( N = \times_{t=1}^{n} N_t \) such that for some \( k, m \in \{1, 2, \ldots, n\} \), \( N_k \) is a classical prime submodule of \( M_k \), \( N_m \) is a classical prime submodule of \( M_m \), and \( N_t = M_t \) for every \( t \in \{1, 2, \ldots, n\} \backslash \{k, m\} \).

Proof. We argue induction on \( n \). For \( n = 2 \) the result holds by Theorem 7. Then let \( 3 \leq n < \infty \) and suppose that the result is valid when \( K = M_1 \times \cdots \times M_{n-1} \). We show that the result holds when \( M = K \times M_n \). By Theorem 7, \( N \) is a classical 2-absorbing submodule of \( M \) if and only if either \( N = L \times M_n \) for some classical 2-absorbing submodule \( L \) of \( K \) or \( N = K \times L_n \) for some classical 2-absorbing submodule \( L_n \) of \( M_n \) or \( N = L \times L_n \) for some classical prime submodule \( L \) of \( K \) and some classical prime submodule \( L_n \) of \( M_n \). Notice that by Lemma 1, a proper submodule \( L \) of \( K \) is a classical prime submodule of \( K \) if and only if \( L = \times_{t=1}^{n-1} N_t \) such that for some \( k \in \{1, 2, \ldots, n-1\} \), \( N_k \) is a classical prime submodule of \( M_k \), and \( N_t = M_t \) for every \( t \in \{1, 2, \ldots, n-1\} \backslash \{k\} \). Consequently we reach the claim.

References


