**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 10, No. 4, 2017, 858-870
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

# New derived systems of Hide's coupled dynamo model

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**Abstract.** In this paper, we present the results of a preliminary analytical and numerical study of new derived system of single and double coupled self-exciting Faraday disk homopolar dynamos by Hide et al [3, 7]. Also, well-known systems including, Rikitake and Bullard systems have been reached by using Tikhonov theorem [6, 10] and eliminating method.

**2010 Mathematics Subject Classifications**: 37C45, 37C25

**Key Words and Phrases**: Dynamo model, Dynamical system, Bifurcations, Tikhonov's theorem, Stability

## 1. Introduction

Hide *et al* [3] have studied the novel autonomous sets of dimensionless nonlinear ordinary differential equations (ODEs)

$$\begin{aligned} \dot{x} &= x(y-1) - \beta z, \\ \dot{y} &= \alpha(1-x^2) - \kappa y, \\ \dot{z} &= x - \lambda z, \quad \textit{where} \quad \dot{x} = dx/d\tau, \quad \textit{etc.} \end{aligned} \tag{1}$$

These equations govern the behaviour of self-exciting homopolar dynamo system. The independent variable $\tau$ denotes time $t$. The dependent variables are $x(\tau), y(\tau)$ and $z(\tau)$ such that $x(\tau)$ is the rescaled electric current in the dynamo, $y(\tau)$ is the angular rotation rate of the disk and $z(\tau)$ measures the angular speed of rotation of the motor. Also, we have four parameters $(\alpha, \beta, \kappa, \lambda)$ which are the system dependents on. These four parameters must be positive because they are physically unrealistic otherwise. Parameters represent where $\alpha$ measures the applied couple; $\beta$ measures the inverse moment of inertia of the armature; $\kappa$ measures the mechanical friction in the disk and $\lambda$ measures the mechanical friction in the motor.

Hide [7] has introduced a system of $N$ self-exciting Faraday disk homopolar dynamos, symmetrically coupled, arranged in a ring. Each unit has an electric motor and is connected in series with a coil and a disk, being driven into motion by the dynamo. The

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system (1) study the case when $N = 1$ (one coupled dynamo). Hide [2] has extended the above study and consider the $N = 2$ case (two coupled dynamos) with their six dependent variables and thirteen dimensionless parameters in general. The system where $N = 2$ is given by the following set of nonlinear ordinary differential equations [7]:

$$\begin{aligned}
\dot{x}_1 &= m x_2 y_1 - x_1 - \beta z_1, \\
\dot{y}_1 &= \alpha(1 - m x_2 x_1) - \kappa y_1, \\
\dot{z}_1 &= x_1 - \lambda z_1. \\
\dot{x}_2 &= l^{-1}[x_1 y_2 - r x_2 - h\beta z_2], \\
\dot{y}_2 &= a^{-1}[\alpha(g - x_2 x_1) - k\kappa y_2], \\
\dot{z}_2 &= b^{-1}[h x_2 - d\lambda z_2].
\end{aligned} \tag{2}$$

In this paper, we derive new systems from $N = 2$ coupled dynamo models by applying some restrictions in the parameters. These restrictions allow us to simplify the system (2) and derive new reduced systems, which may find interested systems or lead to one of the well-known systems. We set all addition parameters to unity unless $a, b$ and $l$. In other words, we consider some constraints that being used

$$k = r = m = h = d = g = 1. \tag{3}$$

Note that when $a = b = l = 1$, we turn to Hide's single-disk homopolar dynamo (1).

Also, we apply an important theorem in perturbation theory that so-called *Tikhonovs Theorem* which reduces the dimension of system. The idea of the theorem is based on singular perturbation problem [8]. Consider a system of differential equations of the form

$$\begin{aligned}
\dot{x} &= f(x, z, t), \\
\mu\dot{z} &= g(x, z, t), \quad 0 < \mu \ll 1
\end{aligned} \tag{4}$$

where $f$ and $g$ are sufficiently differentiable. The functions $f, g$ and the initial values $x(0), z(0)$ may depend smoothly on $\mu$. For simplicity of notation we suppress this dependence. The corresponding equation for $\mu = 0$,

$$\begin{aligned}
\dot{x} &= f(x, z, t), \\
0 &= g(x, z, t),
\end{aligned} \tag{5}$$

is *the reduced problem.* The function $f$ is supposed to possess an isolated solution $z = \xi(x, t)$. The substituting of $z$ into the first equation yields to

$$\dot{x} = f(x, \xi(x, t), t), \quad x(0) = x_0. \tag{6}$$

The paper is structured as follows. In the section 2, we provide an overview of Single-Disk homopolar dynamo that was given by Hide et al [3]. We also present The Double-Disk homopolar dynamo and study. In Double-Disk homopolar dynamo [7], we have case of study, when $\kappa = 0$ in section 3. Throughout this case, we attempt to reduce the dimension of system which make it simple and comparable with well-known system. Numerical analysis including numerical integration, bifurcations study and linear stability are included. We draw the conclusion in section 4.

## 2. Theory

### 2.1. Single-Disk homopolar dynamo: Theory

Hide *et al.* [3] proposed a model for self-exciting dynamo action in which a Faraday disk and coil are arranged in series with either a capacitor or a motor. The system (1) contains a steady equilibrium solution either

$$(x, y, z) = (0, \alpha/\kappa, 0) \quad \text{or} \quad (x, y, z) = (\pm\sqrt{1 - \kappa/\alpha(\beta/\lambda + 1)}, \beta/\lambda + 1, x/\lambda).$$

A linear stability analysis of (1) about the steady equilibrium solution shows that the eigenvalues of Jacobian matrix are

$$-\kappa, \quad 1/2\{\alpha/\kappa - 1 - \lambda \pm \sqrt{(\alpha/\kappa - 1 + \lambda)^2 - 4\beta}\}$$

Steady bifurcations occur along the line

$$\alpha/\kappa = \beta/\lambda + 1,$$

where symmetry breaking bifurcation occur and two lines of Hopf bifurcations along

$$\alpha/\kappa = \lambda + 1,$$

provided $\beta \geq \lambda^2$ and

$$\alpha/\kappa = [(2\beta - \kappa\lambda - \lambda^2)/2(\kappa - \beta/\lambda) + 3\beta/2\lambda + 1].$$

In bifurcation diagram ( see Fig.5 in [3]), there is a Taken-Bogdanov (double zero eigenvalue type) bifurcations occur at the point $P$ where

$$P = (\alpha/\kappa, \beta) = (\lambda + 1, \lambda^2),$$

Note that, all lines of steady bifurcations and two lines of Hopf bifurcations meet in at point $P$ with reflection symmetry. There is a global bifurcation occurs and it emerges from the Taken-Bogdanov point, label it $P$ in bifurcation diagram. In addition Hide *et al.* [3] have shown that dynamo action occurs when the steady equilibrium solution $(x, y, z) = (0, \alpha/\lambda, 0)$ is unstable, namely when $\alpha/\kappa > \min(1 + \beta/\lambda, 1 + \lambda)$, but not otherwise.

### 2.2. Double-disk homopolar dynamo: Theory

The set of equations for general coupled dynamo (2) is reduced by a constrains (3) to:

$$\begin{aligned}
\dot{x}_1 &= x_2y_1 - x_1 - \beta z_1, \\
\dot{y}_1 &= \alpha(1 - x_2x_1) - \kappa y_1, \\
\dot{z}_1 &= x_1 - \lambda z_1. \\
\dot{x}_2 &= l^{-1}[x_1y_2 - x_2 - \beta z_2], \\
\dot{y}_2 &= a^{-1}[\alpha(1 - x_2x_1) - \kappa y_2], \\
\dot{z}_2 &= b^{-1}[x_2 - \lambda z_2].
\end{aligned} \tag{7}$$

Similar to the single homopolar dynamo, the equations (7) have a reflection symmetry: they are unchanged and invariant under the transformation

$$(x_1, y_1, z_1, x_2, y_2, z_2) \to (-x_1, y_1, -z_1, -x_2, y_2, -z_2).$$

In other word, the $y_1, y_2$ are invariant. The equilibrium states can be found in Ref. [4].

## 3. Case Study

The case is when we have nonzero $\lambda$ and zero $\kappa$. The models reduction here are based on two kind of reduction methods. First of all, eliminating the differential equation. Second of all, we use an important theorem in *perturbation theory* [6, 8, 10] that so-called *Tikhonov'sTheorem*. We start the problem in the system (7)with $\kappa = 0$, and hence

$$\begin{aligned} \dot{x}_1 &= x_2 y_1 - x_1 - \beta z_1, \\ \dot{y}_1 &= \alpha(1 - x_2 x_1), \\ \dot{z}_1 &= x_1 - \lambda z_1. \\ \dot{x}_2 &= l^{-1}[x_1 y_2 - x_2 - \beta z_2], \\ \dot{y}_2 &= a^{-1}[\alpha(1 - x_2 x_1)], \\ \dot{z}_2 &= b^{-1}[x_2 - \lambda z_2]. \end{aligned} \tag{8}$$

First of all, we reduce the six dimensions of system (8) to five dimensions by eliminating the equation of $\dot{y}_2$ with $\dot{y}_1$. Suppose

$$\dot{y}_1 = f(x_1, x_2),$$

and so,

$$\dot{y}_2 = a^{-1} f(x_1, x_2).$$

Let

$$\dot{v} = f(x_1, x_2),$$

and hence

$$\begin{aligned} \dot{y}_2 &= a^{-1}\dot{v}, \\ y_2 &= a^{-1}v + c, \end{aligned}$$

Thus, the relation between $y_1$ and $y_2$ is

$$y_2 = a^{-1} y_1 + c. \tag{9}$$

From the equation (9), we can find the simple conditions on parameters such that $a y_2(t) \to y_1(t)$ as $t \to \infty$, if $\kappa = 0$ and the new variable $c = 0$. It is clear that if we differentiate the equation (9), we get $\dot{c} = 0$. In other word, if the solution of $y_2$ in six dimensional system is stable (unstable), then the solution of $y_1$ in five dimensional system is also stable (unstable), respectively. Thus the system (8) becomes

$$\begin{aligned} \dot{x}_1 &= x_2 y_1 - x_1 - \beta z_1, \\ \dot{y}_1 &= \alpha(1 - x_2 x_1), \\ \dot{z}_1 &= x_1 - \lambda z_1. \\ \dot{x}_2 &= l^{-1}[x_1(a^{-1} y_1 + c) - x_2 - \beta z_2], \\ \dot{z}_2 &= b^{-1}[x_2 - \lambda z_2]. \end{aligned} \tag{10}$$

This system can be reduced further. We have five dimensional system (10), and by rescaling $x_1$ to be $x_1 = \lambda X_1$, the $\dot{z}_1$ equation as result becomes

$$\dot{z}_1 = \lambda X_1 - \lambda z_1.$$

Divide it by $\lambda$

$$\dot{z}_1/\lambda = X_1 - z_1.$$

Now we apply Tikhonov Theorem by choosing $\lambda$ large, and hence

$$X_1 = z_1. \tag{11}$$

Similarly in $\dot{z}_2$, we rescale $x_2$ as $x_2 = \lambda X_2$ and then divide the equation by $\lambda$, we get

$$b/\lambda \dot{z}_2 = X_2 - z_2.$$

Set $b = c_1\lambda$. The equation becomes $c_1\dot{z}_2 = X_2 - z_2$, then apply Tikhonov Theorem by choosing $c_1$ to be small ($c_1 \ll 1$). Thus

$$X_2 = z_2. \tag{12}$$

However, this is not always true, it can be work in term of perturbation theory. From (11), (12) and already $y_2$ has been eliminated with $y_1$, the 5D system (10) reduces to 3D system as follows

$$\begin{aligned} \dot{X}_1 &= X_2 y_1 - (1 + \beta/\lambda)X_1, \\ \dot{y}_1 &= \alpha(1 - \lambda^2 X_1 X_2), \\ \dot{X}_2 &= l^{-1}[X_1(a^{-1}y_1 + c) - (1 + \beta/\lambda)X_2], \end{aligned} \tag{13}$$

Assume that $(1 + \beta/\lambda) = \mu$ as a new parameter and rewrite the system(13) in $x, y$ and $z$ coordinates

$$\begin{aligned} \dot{x} &= zy - \mu x, \\ \dot{y} &= \alpha(1 - \lambda^2 xz), \\ \dot{z} &= l^{-1}[x(a^{-1}y + c) - \mu z], \end{aligned} \tag{14}$$

One of the interesting point that Rikitake disc dynamo [9] is involved in the system (14) by setting the limit of parameters as

$$c = -\gamma \quad and \quad l = a = \alpha = \lambda = 1.$$

Hence we have the Rikitake system

$$\begin{aligned} \dot{x} &= zy - \mu x, \\ \dot{y} &= x(z - \gamma) - \mu y, \\ \dot{z} &= 1 - xy, \end{aligned} \tag{15}$$

At value of parameters $\mu = 1.1$ and $\gamma = 7$, we have found that system has chaotic oscillations as shown in figures 1.

In more details, Cook and Roberts [1] found the fixed points of the Rikitake system which are $N$ and $R, (\pm K, \pm K^{-1}, \mu K^2)$ where $K$ is given by

$$\gamma = \mu(K^2 - K^{-2}),$$

or

$$K = \sqrt{\frac{\gamma \pm \sqrt{\gamma^2 + 4\mu^2}}{2\mu}}.$$

There are a stable and centre manifold through each of the fixed points [5]. Thus the dynamics take place on centre manifold, on which the fixed points are both unstable. The stability of the fixed points $R, N$ can be determined by the method of the eigenvalues. In the following presentation we will consider the fixed point $R$ that has positive sign but the same computation may be applied to $N$. If we compute the Jacobian of the system (15) and evaluate it at $R$, we get the matrix

$$J(R) = \begin{bmatrix} -\mu & \mu K^2 & K^{-1} \\ \mu K^2 - \gamma & -\mu & K \\ -K^{-1} & -K & 0 \end{bmatrix}$$

Compute the eigenvalues of the this matrix results in characteristic equation of the form

$$(\sigma^2 + K^2 + K^{-2})(\sigma + 2\mu) = 0,$$

which has roots

$$\sigma_0 = -2\mu, \quad \sigma_{1,2} = \pm(K^2 + K^{-2})i.$$

Since the system (15) has two purely imaginary roots then the eigenvalues method fails. There are two options available to study. Firstly, take the higher order Taylor approximation of system

$$\dot{\xi}(t) = f_\mu(\xi(t) + u^*)$$

for system $\dot{u} = f_\mu(u)$ where $\mu$ parameter and $u^*$ state-solution. Moreover, $\xi(t)$ is the function represent the distance between the state solution and some other solution as

$$\xi(t) = u(t) - u^* \quad with \quad u(0) - u^* = \xi_0,$$

Note that the system that results from such a truncation is no longer linear. Thus it fails too. The other option, we can use the Lyapounov function. This approach is more applicable, to show that points $R$ and $N$ are both unstable for any value of parameters and $K$ saves when $K = 1$ or $\mu = 0$.

In general, the Rikitake system does not possess closed form solution. For certain parameter values the analytical solutions are known. So if we choose the values as $K = 1$ and $\mu = 0$ for the system (15), then we have the following system

$$\begin{aligned} \dot{x} &= zy, \\ \dot{y} &= 1 - xz, \\ \dot{z} &= xy, \end{aligned}$$

If we consider those solution for which $x = z$ and perform the substitution $u = x = z$, then we get the following system

$$\begin{aligned} \dot{u} &= zu, \\ \dot{y} &= 1 - u^2, \end{aligned} \tag{16}$$

These equations are well-known which are called the Bullard model dynamo [2]. Thus, two well-known systems (Rikitake [9] and Bullard [2]) have been derived from original system (2).

Back to the five dimensional system (10) to study the linear stability around it fixed points. By setting all differential equations in (10) equal to zero, we get the fixed point. Firstly, from third and fifth equation, it is easy to see that $z_1 = \frac{1}{\lambda}x_1$ and $z_2 = \frac{1}{\lambda}x_2$, respectively. Also in the second equation we find

$$x_1 x_2 = 1. \tag{17}$$

Secondly, the first and fourth equations can be studied as follows, in the first equation, we have

$$\begin{gathered} x_2 y_1 - (1 + \beta/\lambda)x_1 = 0 \\ x_1 = \frac{x_2 y_1}{\mu} \quad where \quad \mu = 1 + \beta/\lambda. \end{gathered} \tag{18}$$

Also in the fourth equation, we have

$$x_2 = \frac{1}{\mu}(a^{-1}x_1 y_1 + c x_1)$$

Substitute (18) in $x_2$ we get

$$x_2 = \frac{1}{\mu}((a\mu)^{-1}x_2 y_1^2 + \frac{c}{\mu}x_2 y_1)$$

$$Ay_1^2 + By_1 - 1 = 0$$

where $A = \frac{1}{a\mu^2}$ and $B = \frac{c}{\mu}$. Hence

$$y_1 = \frac{-B \pm \sqrt{B^2 - 4A}}{2A} \quad \Rightarrow \quad y_1 = \frac{\frac{-c}{\mu} \pm \sqrt{\frac{c^2}{\mu^2} - \frac{4}{a\mu^2}}}{\frac{2}{a\mu^2}}$$

Rearrange the fraction of $y_1$

$$y_1 = \frac{-c \pm \sqrt{c^2 - \frac{4}{a}}}{2/a\mu}.$$

However, if we multiply (18) by $x_1$ and use the relation in (17), then it simplifies to become $y_1 = \mu x_1^2$ which means that $y_1$ is always positive. Hence we neglect the negative sign of the value of $y_1$. Now plug that in the equation (17), so we have

$$x_1 = \underbrace{(\frac{-c \pm \sqrt{c^2 - \frac{4}{a}}}{2/a\mu})}_{H} x_2,$$

then substitute it in (17)

$$Hx_2^2 = 1 \quad \Rightarrow \quad x_2 = \pm\frac{1}{\sqrt{H}}.$$

Similarly

$$x_1 = \pm\sqrt{H}.$$

As a result the fixed point states are given by:

$$x_1 = \pm\sqrt{H}, \quad x_2 = \pm\frac{1}{\sqrt{H}}, \quad y_1 = \mu H, \quad z_1 = \frac{\sqrt{H}}{\lambda} \quad z_2 = \frac{1}{\lambda\sqrt{H}}, \tag{19}$$

where $H = (\frac{-c\pm\sqrt{c^2-\frac{4}{a}}}{2/a\mu})$ and $\mu = 1 + \beta/\lambda$.

To study the stability of (10) we need to compute the Jacobian matrix of the system (10). Thus, we get

$$J = \begin{bmatrix} -1 & x_2 & -\beta & y_1 & 0 \\ -\alpha x_2 & 0 & 0 & -\alpha x_1 & 0 \\ 1 & 0 & -\lambda & 0 & 0 \\ l^{-1}(a^{-1}y_1 + c) & x_1(al)^{-1} & 0 & -l^{-1} & -l^{-1}\beta \\ 0 & 0 & 0 & b^{-1} & -b^{-1}\lambda \end{bmatrix} \tag{20}$$

Since we have seen a good result in Rikitake equations in figure 1 when $\mu = 1.1$ and $\gamma = 7$, we are going to apply the same values of parameters to find the relationship between the system (10) and Rikitake systems. However the limit of these parameters values come form [7] where assumed $K = 2$ and a value of $\mu$ between 1 and 2 which Rikitake model would the observed reversals are most faithfully. The relation occurs when $c = -\gamma$ and $l = a = \alpha = \lambda = 1$ and keep the value of $b$ as small value, to be 0.001. Since $\mu = 1.1$ and $\mu = 1 + \beta/\lambda$, then $\beta = 0.1$. Before substituting fixed points in the Jacobian matrix, we need to calculate the value of $x_1, x_2, y_1$ which are $\pm 2.55, \pm 0.39$ and 7.16, respectively. So the Jacobian matrix at the fixed points is

$$J(fp) = \begin{bmatrix} -1 & \pm 0.39 & -0.1 & 7.16 & 0 \\ \mp 0.39 & 0 & 0 & \mp 2.55 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0.16 & \pm 2.55 & 0 & -1 & -0.1 \\ 0 & 0 & 0 & 1000 & -1000 \end{bmatrix} \tag{21}$$

Thus the eigenvalues are as follows

```
-9.998998904039592e+002
 5.863307789355593e-003 +2.578916992181417e+000i
 5.863307789355593e-003 -2.578916992181417e+000i
-2.030767974296189e+000
-1.081068237323492e+000
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Since we have three negative eigenvalues and a pair of complex conjugate eigenvalues with positive real part, this gives an indication of having unstable centre (foci) equilibrium [5] as in Rikitake equilibrium points.

The parameters values as those in Rikitake $\Gamma = \mu = 1.1$ , $c = -\gamma = -7$ and $l = a = \alpha = 1$ and keep the value of $b$ as small value, to be 0.001 to show the phase portrait in system (10). It would be shown in figures 2 and 3, respectively.

It shows that the solution of $x_2$ and $z_2$ is clearly having same behaviour. It also shows that phase portrait in the $(x_2, z_2)$, $(x_1, x_2)$, $(z_1, z_2)$, $(z_1, z_2)$. $(x_2, y_1)$, $(x_1, z_2)$ and $(x_2, z_1)$ are indicated of chaos behaviour. The behaviours of $(x_1, y_1)$ are similar to the behaviours of $(x, y)$ in Rikitike system. Also, the behaviours of $(x_2, y_1)$ or $(z_2, y_1)$ are similar to the behaviours of $(z, y)$ in Rikitike system. Moreover, the behaviours of $(x_1, x_2)$ and $(x_1, z_2)$ are similar. The behaviours of $(z_1, x_2)$ and $(z_1, z_2)$ are similar. Also, the behaviours of $(y_1, z_2)$ and $(y_1, x_2)$ are similar.

## 4. Conclusion

The system of a double disk dynamo with motors is valuable and many cases can be studied to see the beauty of the dynamical system including, Chaos, Bifurcations and periodic orbits. In our case, we show how well-known system can derived from the original system using the reduction method provided.

A further cases can be studied be setting the parameters to investigate many possibility in the dynamical system theory.

## Acknowledgements

The author is supported by Qassim University and the Ministry of Education of Saudi Arabia.

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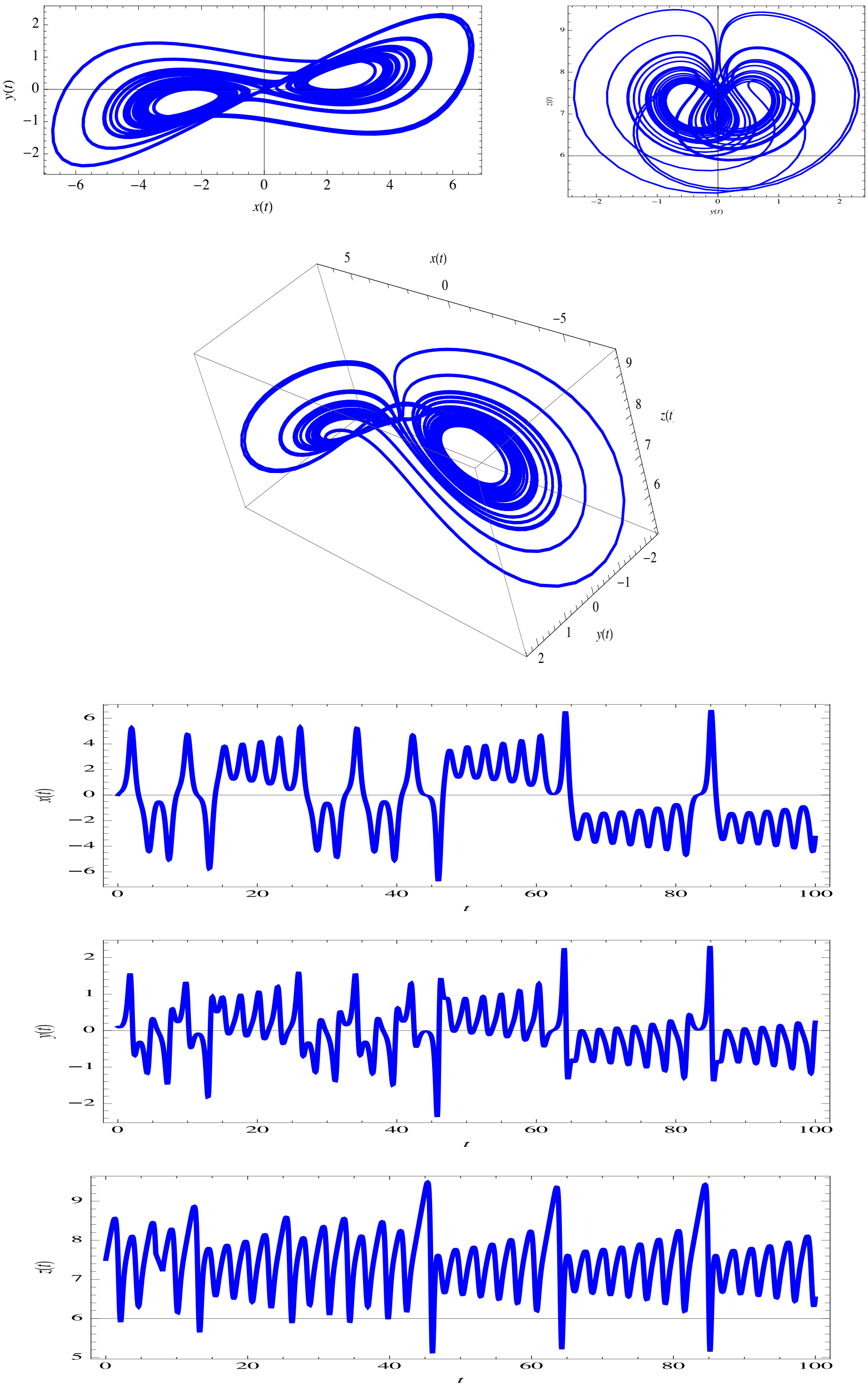


Figure 1: Rikitake system with parameters choice as $\mu = 1.1$ and $\gamma = 7$.

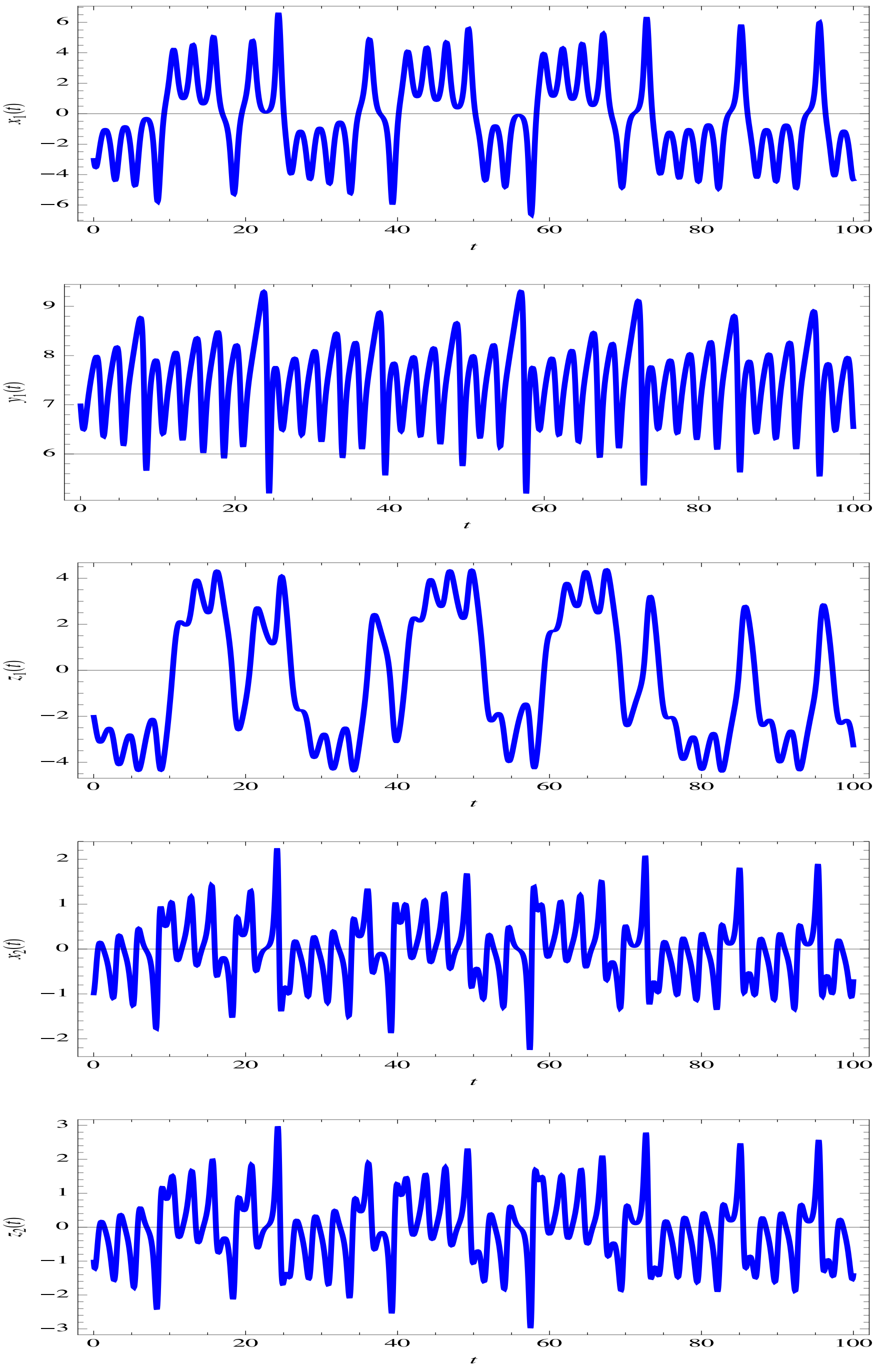


Figure 2: Five dimensional system with parameters choice as $\gamma = 7, \alpha = 1, \beta = 0.0001, l = 1, c = -7, \kappa = 0.60, \lambda = 0.65, b = 0.1, a = 1$, and $\mu = 1.1$.

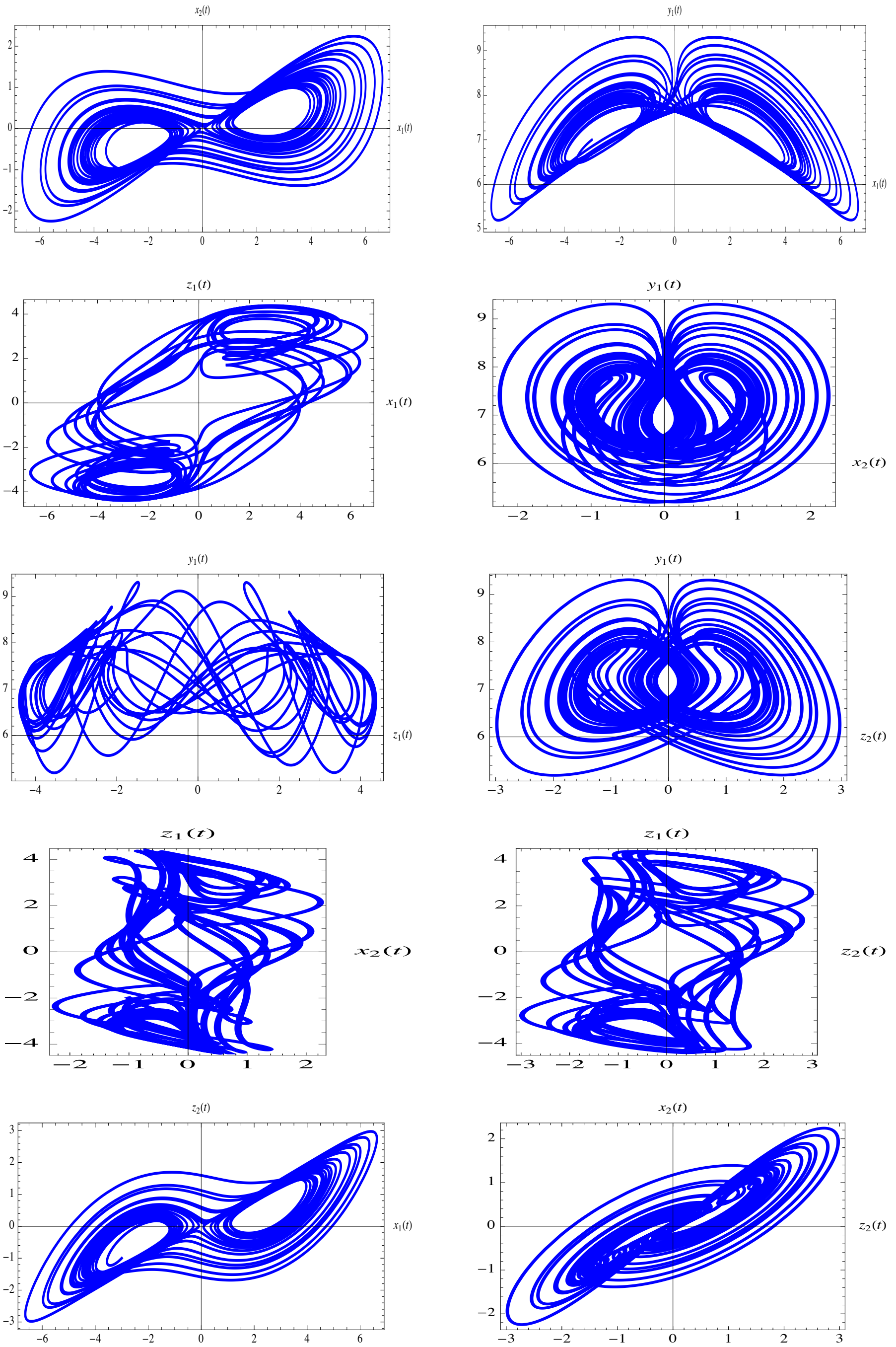


Figure 3: Continued five dimensional system with parameters choice as $\gamma = 7, \alpha = 1, \beta = 0.0001, l = 1, c = -7, \kappa = 0.60, \lambda = 0.65, b = 0.1, a = 1$, and $\mu = 1.1$.