On $\beta$-Open Sets and Ideals in Topological Spaces

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Abstract. Let $X$ be a topological space and $I$ be an ideal in $X$. A subset $A$ of a topological space $X$ is called a $\beta$-open set if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. A subset $A$ of $X$ is called $\beta$-open with respect to the ideal $I$, or $\beta_I$-open, if there exists an open set $U$ such that (1) $U - A \in I$, and (2) $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. A space $X$ is said to be a $\beta_I$-compact space if it is $\beta_I$-compact as a subset. An ideal topological space $(X, \tau, I)$ is said to be a $c\beta_I$-compact space if it is $c\beta_I$-compact as a subset. An ideal topological space $(X, \tau, I)$ is said to be a countably $\beta_I$-compact space if $X$ is countably $\beta_I$-compact as a subset. Two sets $A$ and $B$ in an ideal topological space $(X, \tau, I)$ is said to be $\beta_I$-separated if $\text{cl}_{\beta_I}(A) \cap B = \emptyset = A \cap \text{cl}_{\beta_I}(B)$. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta_I$-connected if it cannot be expressed as a union of two $\beta_I$-separated sets. An ideal topological space $(X, \tau, I)$ is said to be $\beta_I$-connected if $X$ is $\beta_I$-connected as a subset. In this study, we introduced the notions $\beta_I$-open set, $\beta_I$-compact, $c\beta_I$-compact, $\beta_I$-hyperconnected, $c\beta_I$-hyperconnected, $\beta_I$-connected and $\beta_I$-separated. Moreover, we investigated the concept $\beta$-open set by determining some of its properties relative to the above-mentioned notions.

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1. Introduction

Topology is an interesting area of mathematics. It is new, being conceive in the 19th century. But according to Morris [15], the influence of topology is so vast, so that it is identifiable in various branches of mathematics.

Topological ideas are present not only in mathematics but also in other areas, for example biochemistry [16] and information systems [17]. Topology as a subject has several different branches such as point set topology, algebraic topology, differential topology, etc.

The basic component of a topology space are open sets, and overtime there have been so many generalizations of it. Among them are the following. Stone [18] introduced
the concept of regular open sets. Levine [19] introduced the concept of semi open sets. Najasted [20] introduced the concept of $\alpha$-open sets. Mashhour et al. [31] introduced the concept of pre-open sets. Abd El-Monsef et al. [1] introduced the concept of $\beta$-open sets.

Apart from introducing $\beta$-open sets, Abd El-Monsef et al. [1] also introduced $\beta$-continuous mappings and $\beta$-open mappings. They studied their properties and discussed the connections of these notions with the existing ones. Since then, the concept $\beta$-open sets has been a subject of a couple of investigations. Among them were the following. Abid [22] used the concept $\beta$-open set to obtain the properties of the concept non-semi-predense set. Tahiliani [23] introduced an operation on a family of $\beta$-open sets; and using the operation, the concept $\beta$-$\gamma$-open sets was defined and investigated. Kannan and Nagaveni [5] introduced another generalization of the concept $\beta$-open sets, called $\beta$-generalized closed sets. Mubarki et al. [25] introduced and investigated $\beta^*$-open sets, which is also a generalization of the concept $\beta$-open sets. El-Mabhouh and Mizyed [26] introduced the concept $\beta$-closed set which is a particular class of $\beta$-open sets. They also showed that $\beta$-c-open sets generates the same topology as the class of $\theta$-open sets in Alexandroff space. Akdag and Ozkan [27] adapted the concept $\beta$-open set in soft topological spaces, and defined the concepts soft $\beta$-interior and soft $\beta$-closure, and gave their properties. Arockiarrani and Arokiar Lancy [28] presented $g\beta$-closed sets and $gs\beta$-closed sets (which were defined indirectly in terms of the notion of $\beta$-open sets) and introduced parallel concepts in soft topological spaces.

Let $X$ be a topological space and $I$ be an ideal in $X$. A subset $A$ of a topological space $X$ is called a $\beta$-open set if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. A subset $A$ of $X$ is called $\beta$-open with respect to the ideal $I$, or $\beta_I$-open, if there exists an open set $U$ such that (1) $U - A \in I$, and (2) $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta_I$-compact if every cover of $A$ by $\beta_I$-open set has a finite sub-cover. A space $X$ is said to be a $\beta_I$-compact space if it is $\beta_I$-compact as a subset. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be a compatible $\beta_I$-compact, or simply $c\beta_I$-compact, if every cover $\{U_\lambda : \lambda \in \Lambda\}$ of $A$ by $\beta$-open set has a finite subset $\Lambda_0$ of $\Lambda$ such that $A - \bigcup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space $(X, \tau, I)$ is said to be a $c\beta_I$-compact space if it is $c\beta_I$-compact as a subset.

A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be countably $\beta_I$-compact if every countable cover $\{U_n : n \in \mathbb{N}\}$ of $A$ by $\beta_I$-open set, there exists a finite subset $\{i_1, i_2, \ldots, i_k\}$ of $\mathbb{N}$ such that $A - \bigcup\{U_{i_j} : j = 1, 2, \ldots, k\} \in I$. An ideal topological space $(X, \tau, I)$ is said to be a countably $\beta_I$-compact space if $X$ is countably $\beta_I$-compact as a subset.

The concept $*-\text{hyperconnectedness}$ was introduced by Ekici et al. [2], and the concept $I*-\text{hyperconnectedness}$ was introduced by Abd El-Monsef et al. [7]. As defined in [4] an ideal topological space $(X, \tau, I)$ is said to be $*-\text{hyperconnected}$ if $\text{cl}^*(A) = X$ for every non-empty open subset $A$ of $X$, and as defined in [3], an ideal topological space $(X, \tau, I)$ is said to be $I*-\text{hyperconnected}$ if $X - \text{cl}^*(A) \in I$ for every non-empty open subset $A$ of $X$. Given these insights, we introduce the following parallel concept. An ideal topological space $(X, \tau, I)$ is said to be $\beta^*_I$-hyperconnected if $X - \text{cl}^*(A) \in I$ for every non-empty $\beta_I$-open subset $A$ of $X$. 

For the concepts that were not discussed here please refer to [6, 14, 15].

2. $\beta$-Open Sets with Respect to an Ideal

In this section, we investigated the concept $\beta$-open in a direction parallel to the investigation of semi-open sets in [32].

**Lemma 1.** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then \(\text{int}(A) = \text{int}(\text{cl}(\text{int}(A)))\).

**Proof.** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then we have, \(\text{int}(\text{cl}(\text{int}(A))) = \text{int}(\text{Fr}(A) \cup \text{int}(A)) = \text{int}(\text{Fr}(\text{int}(A)) \cup \text{int}(A)) = \text{int}(\text{cl}(\text{int}(A)))\). $\square$

**Lemma 2** characterizes $\beta$-open sets.

**Lemma 2.** Let $(X, \tau, I)$ be an ideal topological space. A subset $A$ of $X$ is $\beta$-open if and only if there exists an open set $U$ such that $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$.

**Proof.** Assume that $A$ is $\beta$-open. Then $A \subseteq \text{cl}(\text{int}(A))$. Let $U = \text{int}(A)$. Then $U$ is open and, by Lemma 1 $U \subseteq A \subseteq \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(\text{int}(A))) = \text{cl}(\text{int}(U))$.

Conversely, assume that there exists an open set $U$ such that $U \subseteq A \subseteq \text{cl}(\text{int}(U))$. Since $U \subseteq A$, $\text{cl}(U) \subseteq \text{cl}(A)$. Hence, $\text{int}(\text{cl}(U)) \subseteq \text{int}(\text{cl}(A))$. And so, $\text{cl}(\text{int}(\text{cl}(U))) \subseteq \text{cl}(\text{int}(\text{cl}(A)))$ Thus, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$. $\square$

**Lemma 3** says that every open set is a $\beta_I$-open set, every element of the ideal is a $\beta_I$-open set, and every $\beta$-open set is a $\beta_I$-open set.

**Lemma 3.** Let $(X, \tau, I)$ be an ideal topological space.

(i) If $A$ is an open set, then $A$ is an $\beta_I$-open set.

(ii) If $A \in I$, then $A$ is an $\beta_I$-open set.

(iii) If $A$ is a $\beta$-open set, then $A$ is an $\beta_I$-open set.

**Proof.** (1) If $A$ is open, then we let $U = A$. Observe that $U - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = A - \text{cl}(U) = A - \text{cl}(A) = \emptyset \in I$. This shows that $A$ is an $\beta_I$-open set. (2) If $A \in I$, then we let $U = \emptyset$. Observe that $U - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = A - \emptyset = A \in I$. This shows that $A$ is an $\beta_I$-open set. (3) If $A$ is $\beta$-open, then by Lemma 2 there exists an open set $U$ such that $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. Observe that $U - A = \emptyset \in I$, and $A - \text{cl}(\text{int}(\text{cl}(U))) = \emptyset \in I$. This shows that $A$ is an $\beta_I$-open set. $\square$

**Lemma 4** says that if $I$ is the minimal ideal, then the $\beta_I$-open sets are precisely the $\beta$-open sets.
Lemma 4. Let $(X, \tau, I)$ be an ideal topological space. If $I$ is not countably additive, then the following statements are equivalent.

(i) If $I = \{ \emptyset \}$.

(ii) $A$ is a $\beta$-open set if and only if $A$ is a $\beta_I$-open set.

Proof. Assume that $I = \{ \emptyset \}$, and $A$ be a $\beta$-open set. Then by Lemma 3, $A$ is a $\beta_I$-open set. Conversely, let $A$ be a $\beta_I$-open set. Then there exists an open set $U$ such that $U - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. Since $\emptyset, U - A \in \emptyset$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in \emptyset$, that is, $U \subseteq A$ and $A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. Thus, $U \subseteq A \subseteq \text{cl}(\text{int}(\text{cl}(U)))$. By Lemma 2, $A$ is a $\beta$-open set.

Next, assume that $A$ is a $\beta$-open set if and only if $A$ is a $\beta_I$-open set, and suppose that $I \neq \{ \emptyset \}$. Let $B \in I$ with $B \neq \emptyset$. Then by Lemma 3, $B$ is a $\beta_I$-open set. By assumption $B$ is a $\beta$-open set. By Lemma 2, there exists an open set $U_1$ such that $U_1 \subseteq B \subseteq \text{cl}(\text{int}(\text{cl}(U_1)))$. Since $B \in I$ and $U_1 \subseteq B$, $U_1 \in I$. Hence, $U_1 \cup B \in I$. By Lemma 2, $U_1 \cup B$ is a $\beta_I$-open set. By assumption $U_1 \cup B$ is a $\beta$-open set. Thus, there exists an open set $U_2$ such that $U_2 \subseteq (U_1 \cup B) \subseteq \text{cl}(\text{int}(\text{cl}(U_2)))$. Since $U_1 \cup B \in I$ and $U_2 \subseteq U_1 \cup B$, $U_2 \in I$. Hence, $U_1 \cup U_2 \cup B \in I$. By Lemma 2, $U_1 \cup U_2 \cup B$ is a $\beta_I$-open set. By assumption $U_1 \cup U_2 \cup B$ is a $\beta$-open set. Thus, there exists an open set $U_3$ such that $U_3 \subseteq (U_1 \cup U_2 \cup B) \subseteq \text{cl}(\text{int}(\text{cl}(U_3)))$. Since $U_1 \cup U_2 \cup B \in I$ and $U_3 \subseteq U_1 \cup U_2 \cup B$, $U_3 \in I$. Hence, $U_1 \cup U_2 \cup U_3 \in I$. Continuing in this manner we obtain a sequence $(U_1, U_2, U_3, \ldots)$ of set in $I$ such that $U_1 \cup U_2 \cup U_3 \cup \cdots \in I$. This is a contradiction since $I$ is not countably additive. Therefore, $I = \{ \emptyset \}$. \hfill $\Box$

Theorem 1 says that if $I$ is the minimal ideal, then the notions $\beta$-compact, $\beta_I$-compact and $c\beta_I$-compact coincides.

Theorem 1. For an ideal topological space $(X, \tau, I)$, the following statements are equivalent.

(i) $(X, \tau)$ is a $\beta$-compact space.

(ii) $(X, \tau, \{ \emptyset \})$ is a $\beta_I$-compact space.

(iii) $(X, \tau, \{ \emptyset \})$ is a $c\beta_I$-compact space.

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds, and let $\{ U_\lambda : \lambda \in \Lambda \}$ be a cover of $X$ by $\beta$-open sets. By Lemma 3 $U_\lambda$ is a $\beta_I$-open set for all $\lambda \in \Lambda$. Since $X$ is $\beta$-compact, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $\{ U_\lambda : \lambda \in \Lambda_0 \}$ is still a cover of $X$. By Lemma 4, $U_\lambda$ is a $\beta_I$-open set for all $\lambda \in \Lambda_0$. Hence, there exists a finite subcover of $X$ by $\beta_I$-open sets. This shows that $X$ is a $\beta_I$ compact space.

(2) $\Rightarrow$ (3) Assume that (2) holds, and let $\{ U_\lambda : \lambda \in \Lambda \}$ be a cover of $X$ by $\beta$-open sets. By Lemma 3 $U_\lambda$ is a $\beta_I$-open set for all $\lambda \in \Lambda$. By assumption, there exists a finite subset
\[ \Lambda_0 \text{ of } \Lambda \text{ such that } \{ U_\lambda : \lambda \in \Lambda_0 \} \text{ is still a cover of } X, \text{ that is } X - \bigcup U_\lambda \in \Lambda = \emptyset \in I. \text{ This show that (3) holds.} \]

(3) \Rightarrow (1) Assume that (3) holds, and let \( \{ U_\lambda : \lambda \in \Lambda \} \) be a cover of \( X \) by \( \beta \)-open sets. By Lemma 4, if \( I = \{ \emptyset \} \), then a \( \beta \)-open sets is precisely a \( \beta I \)-open set. Thus, \( U_\lambda \) is at the same time a \( \beta I \)-open set for all \( \lambda \in \Lambda \). By assumption, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( X - \bigcup \{ U_\lambda : \lambda \in \Lambda_0 \} \in I \), that is \( X - \bigcup \{ U_\lambda : \lambda \in \Lambda_0 \} = \emptyset \) \( \subseteq \bigcup \{ U_\lambda : \lambda \in \Lambda_0 \} \). Hence, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( X \subseteq \bigcup \{ U_\lambda : \lambda \in \Lambda_0 \} \). This show that (1) holds.

Theorem 2 characterizes \( \beta I \)-compact space.

**Theorem 2.** For an ideal topological space \((X, \tau, I)\), the following statements are equivalent.

(i) \((X, \tau, I)\) is a \( \beta I \)-compact space.

(ii) For every family \( \{ F_\lambda : \lambda \in \Lambda \} \) of \( \beta I \)-closed sets such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \), there exist a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda_0 \} = \emptyset \).

**Proof.** (1) \Rightarrow (2) Assume that (1) holds, and let \( \{ F_\lambda : \lambda \in \Lambda \} \) be a family of \( \beta I \)-closed sets such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \). If \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \), then \( \bigcup \{ F_\lambda^C : \lambda \in \Lambda \} = (\bigcap \{ F_\lambda : \lambda \in \Lambda \})^C = X \). Hence, \( \bigcup \{ F_\lambda^C : \lambda \in \Lambda \} \) is a covering of \( X \) by \( \beta I \)-open sets. By assumption there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( \bigcap \{ U_\lambda^C : \lambda \in \Lambda_0 \} = \emptyset \), that is \( \bigcup \{ U_\lambda : \lambda \in \Lambda_0 \} = X \). This show that (1) holds.

Theorem 3 characterizes \( \beta I \)-compact space.

**Theorem 3.** In an ideal topological space \((X, \tau, I)\), the following statements are equivalent.

(i) \((X, \tau, I)\) is a c\( \beta I \)-compact space.

(ii) For every family \( \{ F_\lambda : \lambda \in \Lambda \} \) of \( \beta I \)-closed sets such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \), there exist a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda_0 \} \in I \).

**Proof.** (1) \Rightarrow (2) Assume that (1) holds, and let \( \{ F_\lambda : \lambda \in \Lambda \} \) be a family of \( \beta I \)-closed sets such that \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \). If \( \bigcap \{ F_\lambda : \lambda \in \Lambda \} = \emptyset \), then \( \bigcup \{ F_\lambda^C : \lambda \in \Lambda \} = (\bigcap \{ F_\lambda : \lambda \in \Lambda \})^C = X \). Hence, \( \bigcup \{ F_\lambda^C : \lambda \in \Lambda \} \) is a covering of \( X \) by \( \beta I \)-open sets. By assumption there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( X - \bigcup \{ F_\lambda^C : \lambda \in \Lambda_0 \} \in I \), that is \( \bigcap \{ F_\lambda : \lambda \in \Lambda_0 \} \in I \).

(2) \Rightarrow (1) Assume that (2) holds, and let \( \{ U_\lambda : \lambda \in \Lambda \} \) be a cover of \( X \) by \( \beta I \)-open sets. If \( \{ U_\lambda : \lambda \in \Lambda \} \) is a cover of \( X \) by \( \beta I \)-open sets, that is \( \bigcup \{ U_\lambda : \lambda \in \Lambda \} = X \), then
\[ \bigcap \{ \mathcal{U}_\lambda : \lambda \in \Lambda \} = \bigcup \{ \mathcal{U}_\lambda : \lambda \in \Lambda \} \neq \emptyset. \] By assumption, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( \bigcap \{ \mathcal{U}_\lambda : \lambda \in \Lambda_0 \} \in I \), that is \( X - \bigcup \{ \mathcal{U}_\lambda : \lambda \in \Lambda_0 \} \in I \). This shows that (1) holds. \( \square \)

**Remark 1.** [30] Let \( (X, \tau, I) \) and \( (Y, \sigma, J) \) be ideal topological spaces.

(i) If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is a function, then \( f(I) = \{ f(A) : A \in I \} \) is an ideal in \( Y \).

(ii) If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is an injective function, then \( f^{-1}(J) = \{ f^{-1}(B) : B \in J \} \) is an ideal in \( X \).

We note that a function \( f : (X, \tau, I) \to (Y, \sigma, J) \) is called

(i) \( \beta_I \)-open function if \( f(A) \) is \( \beta_I \)-open in \( Y \) for each \( \beta_I \)-open set \( A \) in \( X \).

(ii) \( \beta_I \)-irresolute function if \( f^{-1}(B) \) is \( \beta_I \)-open in \( X \) for each \( \beta_I \)-open set \( B \) in \( Y \).

(iii) \( \beta_I \)-continuous function if \( f^{-1}(B) \) is \( \beta_I \)-open in \( X \) for each open set \( B \) in \( Y \).

The following Theorems are worth-noticing.

**Theorem 4.** If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is a \( \beta_I \)-irresolute surjective function and \( (X, \tau, I) \) is a \( c\beta_I \)-compact space, then \( (Y, \sigma, J) \) is also a \( c\beta_I \)-compact space.

**Proof.** Let \( \{ \mathcal{U}_\lambda : \lambda \in \Lambda \} \) be a cover of \( Y \) by \( \beta_I \)-open sets. Since \( f \) is a \( \beta_I \)-irresolute surjective function, \( \{ f^{-1}(\mathcal{U}_\lambda) : \lambda \in \Lambda \} \) is a cover of \( X \) by \( \beta_I \)-open sets. Since \( X \) is \( c\beta_I \)-compact, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( X - \bigcup \{ f^{-1}(\mathcal{U}_\lambda) : \lambda \in \Lambda_0 \} \in I \). By Remark 1, \( Y - \bigcup \{ f(f(\mathcal{U}_\lambda)) : \lambda \in \Lambda_0 \} = f(X - \bigcup \{ f^{-1}(\mathcal{U}_\lambda) : \lambda \in \Lambda_0 \}) \in J \). \( \square \)

**Theorem 5.** If \( f : (X, \tau, I) \to (Y, \sigma, J) \) is a \( \beta_I^{-1}(1) \)-open bijective function and \( (Y, \sigma, J) \) is a \( c\beta_I^{-1}(1) \)-compact space, then \( (X, \tau, f^{-1}(J)) \) is also a \( c\beta_I^{-1}(1) \)-compact space.

**Proof.** Let \( \{ \mathcal{U}_\lambda : \lambda \in \Lambda \} \) be a cover of \( X \) by \( \beta_I^{-1}(1) \)-open sets. Since \( f \) is an open bijective function, \( \{ f(\mathcal{U}_\lambda) : \lambda \in \Lambda \} \) is a cover of \( Y \) by \( \beta_I \)-open sets. Since \( Y \) is a \( c\beta_I \)-compact space, there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( Y - \bigcup \{ f(\mathcal{U}_\lambda) : \lambda \in \Lambda_0 \} \in J \), that is \( X - \bigcup \{ \mathcal{U}_\lambda : \lambda \in \Lambda_0 \} = f^{-1}(Y - \bigcup \{ f(\mathcal{U}_\lambda) : \lambda \in \Lambda_0 \}) \in J \). This shows that \( (X, \tau, f^{-1}(J)) \) is a \( c\beta_I^{-1}(1) \)-compact space. \( \square \)

**Theorem 6.** Every \( c\beta_I \)-compact space is also a countably \( \beta_I \)-compact space.

**Proof.** Let \( (X, \tau, I) \) be \( c\beta_I \)-compact space. Let \( \{ \mathcal{U}_n : n \in \mathbb{N} \} \) be a countable cover of \( X \) by \( \beta_I \)-open sets. Since \( X \) is a \( c\beta_I \)-compact space, there exists a finite subset \( \{ i_j : j = 1, 2, \ldots, k \} \) of \( \mathbb{N} \) such that \( X - \bigcup \{ \mathcal{U}_{i_j} : j = 1, 2, \ldots, k \} \in I \). This shows that \( (X, \tau, I) \) is a countably \( \beta_I \)-compact space. \( \square \)
The concept $*$-hyperconnectedness was introduced by Ekici et al. [2], and the concept $I*$-hyperconnectedness was introduced by Abd El-Monsef et al. [7]. These insights propelled us to create a parallel concept called $\beta_I*$-hyperconnectedness, and the investigation in this section is parallel to the investigation of $\alpha$-open sets in [24].

**Theorem 7.** Every $\beta_I*$-hyperconnected space is also an $I*$-hyperconnected space.

**Proof.** Let $(X, \tau, I)$ be a $\beta_I^*$-hyperconnected space and $A$ be an open set. Since $X$ is a $\beta_I^*$-hyperconnected space and every open set is a $\beta_I$-open set, $X - cl^*(A) \in I$. Thus, $(X, \tau, I)$ is also an $I*$-hyperconnected space. \hfill $\Box$

**Lemma 5.** The intersection of any family of ideals on $X$ is an ideal on $X$.

Theorem 8 say that if $I$ is the minimal ideal, then the notions $*$-hyperconnectedness and $I^*$-hyperconnectedness are the same. Moreover, if the topological space is clopen, then the notions $*$-hyperconnectedness, $I^*$-hyperconnectedness and $\beta_I^*$-hyperconnectedness are the same.

**Theorem 8.** Let $(X, \tau, \{\emptyset\})$ be an ideal topological space.

1. If $I = \{\emptyset\}$, then the concepts $*$-hyperconnectedness and $I^*$-hyperconnectedness are equivalent. [3]

2. If $I = \{\emptyset\}$ and every open set is closed, then the concepts $*$-hyperconnectedness, $I^*$-hyperconnectedness and $\beta_I^*$-hyperconnectedness are equivalent.

**Proof.** (1) If $(X, \tau, I)$ is a $\beta_I^*$-hyperconnected space and $A$ is a non-empty open set, then $cl^*(A) = X$. Hence, $X - cl^*(A) = \emptyset \in I$. Thus, $(X, \tau, I)$ is an $I^*$-hyperconnected space.

Conversely, if $(X, \tau, I)$ is an $I^*$-hyperconnected space and $A$ is a non-empty open set, then $X - cl^*(A) \in I$. Since $I = \{\emptyset\}$, $X - cl^*(A) = \emptyset$, that is $cl^*(A) = X$. Thus, $(X, \tau, I)$ is an $*$-hyperconnected space.

(2) Assume that $I = \{\emptyset\}$ and every open set is closed.

**Claim 1.** $A$ is an open set if and only if $A$ is a $\beta$-open set. If $A$ is an open set, then $cl(int(cl(A))) = A$. Hence, $A$ is a $\beta$-open set. Conversely, if $A$ is a $\beta$-open set, then by Lemma 2 there exists an open set $U$ such that $U \subseteq A \subseteq cl(int(cl(U))) = U$, that is $A = U$. Hence, $A$ is an open set. This shows the claim.

**Claim 2.** The concepts $I^*$-hyperconnectedness and $\beta_I^*$-hyperconnectedness are equivalent. If $(X, \tau, I)$ is an $I^*$-hyperconnected space and $A$ is a non-empty open set, then $X - cl^*(A) \in I$. By Claim 1 and Lemma 5, any open set is precisely a $\beta_I$-open set. Hence, $X - cl^*(A) \in I$ for every non-empty $\beta_I$-open set. Thus, $(X, \tau, I)$ is a $\beta_I^*$-hyperconnected space. Conversely, if $(X, \tau, I)$ is a $\beta_I^*$-hyperconnected space and $A$ is a non-empty $\beta_I$-open set, then $X - cl^*(A) \in I$. By Claim 1 and Lemma 5, any open set is precisely a $\beta_I$-open set. Hence, $X - cl^*(A) \in I$ for every non-empty open set. Thus, $(X, \tau, I)$ is an $I^*$-hyperconnected space.

Accordingly, by statement (1) and Claim 2, statement (2) follows. \hfill $\Box$
Theorem 9. If an ideal topological space \((X, \tau, \varnothing)\) is a \(\beta_I^*\)-hyperconnected space, then \(X - \text{cl}^*(A) \in I\) for every non-empty \(\beta\)-open subset \(A\) of \(X\).

Proof. Let \((X, \tau, I)\) be a \(\beta_I^*\)-hyperconnected space and \(A\) is a non-empty \(\beta\)-open set. Since by Lemma 4 every \(\beta\)-open set is a \(\beta_I\)-open set, \(A\) is a non-empty \(\beta_I\)-open set also. If \((X, \tau, I)\) is a \(\beta_I^*\)-hyperconnected space, then \(X - \text{cl}^*(A) \in I\). \(\square\)

Theorem 10 characterizes a \(\beta_I^*\)-hyperconnected space.

Theorem 10. For an ideal topological space \((X, \tau, I)\), the following statements are equivalent.

\[ 
\begin{align*}
(i) & \ X \text{ is a } \beta_I^*\text{-hyperconnected space.} \\
(ii) & \ \text{int}^*(A) \in I \text{ for every proper } \beta_I\text{-closed subset } A \text{ of } X.
\end{align*}
\]

Proof. (1) \(\Rightarrow\) (2) Assume that (1) holds, and let \(B\) be a \(\beta_I\)-closed set. If \(B\) is a \(\beta_I\)-closed set, then \(X - B\) is a \(\beta_I\)-open set. Moreover, if \(B\) is a proper subset, then \(B^C \neq \varnothing\). By assumption, \(\text{int}^*(B) = X - \text{cl}^*(X - B) \in I\).

(2) \(\Rightarrow\) (1) Assume that (2) holds, and let \(A\) be a non-empty \(\beta_I\)-open set. If \(A\) is a non-empty \(\beta_I\)-open set, then \(X - A\) is a proper \(\beta_I\)-open subset of \(X\). By assumption, \(X - \text{cl}^*(A) = X - \text{cl}^*(X - (X - A)) = \text{int}^*(X - A) \in I\). This shows that \(X\) is a \(\beta_I^*\)-hyperconnected space. \(\square\)

4. Separation Notions with Respect to Ideals

Let \((X, \tau, I)\) be an ideal topological space and \(A\) be a subset of \(X\). The \(\beta_I\)-closure of \(A\) is the smallest \(\beta_I\)-closed set containing \(A\), denoted by \(\text{cl}_{\beta_I}(A)\).

Recall that two sets \(A\) and \(B\) in an ideal topological space \((X, \tau, I)\) is said to be \(\beta_I\)-separated if \(\text{cl}_{\beta_I}(A) \cap B = \varnothing = A \cap \text{cl}_{\beta}(B)\), and a subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be \(\beta_I\)-connected if it cannot be expressed as a union of two \(\beta_I\)-separated sets. An ideal topological space \((X, \tau, I)\) is said to be \(\beta_I\)-connected if \(X\) \(\beta_I\)-connected as a subset. A subset \(A\) of an ideal topological space \((X, \tau, I)\) is said to be \(\beta_I\)-connected if it cannot be expressed as a union of two \(\beta_I\)-separated sets. An ideal topological space \((X, \tau, I)\) is said to be \(\beta_I\)-connected if \(X\) \(\beta_I\)-connected as a subset.

Lemma 6. Let \((X, \tau, I)\) be an ideal topological space. If \(A\) and \(B\) are non-empty disjoint subsets of \(X\) such that \(A\) is \(\beta\)-open and \(B\) is \(\beta_I\)-open, then \(A\) and \(B\) are \(\beta_I\)-separated sets.

Proof. Suppose that \(A\) and \(B\) are \(\beta_I\)-separated sets, that is \(\text{cl}_{\beta_I}(A) \cap B \neq \varnothing\) or \(A \cap \text{cl}_{\beta}(B) \neq \varnothing\). Since \(A\) and \(B\) are non-empty disjoint subsets of \(X\), \(A \subseteq B^C\) and \(B \subseteq A^C\). If \(A\) is \(\beta\)-open and \(B\) is \(\beta_I\)-open, then \(A^C\) is \(\beta\)-closed and \(B^C\) is \(\beta_I\)-closed. Hence, \(B^C \cap B \supseteq \text{cl}_{\beta_I}(A) \cap B \neq \varnothing\) or \(A \cap A^C \supseteq A \cap \text{cl}_{\beta}(B) \neq \varnothing\). This is a contradiction. \(\square\)

The next statement, Lemma 7, stressed that every \(\beta_I\)-connected space is connected.
Lemma 7. If an ideal topological space $(X, \tau, I)$ is $\beta_I$-connected, then $(X, \tau)$ is connected.

Proof. Suppose that $X$ is not connected. Let $A$ and $B$ be non-empty disjoint open sets such that $X = A \cup B$. Since every open set is both $\beta$-open and $\beta_I$-open, $A$ and $B$ are both $\beta$-open and $\beta_I$-open. Since $A = B^C$ and $B = A^C$, $A$ and $B$ are also both $\beta$-closed and $\beta_I$-closed. Thus $A = \text{cl}_\beta(A)$ and $B = \text{cl}_\beta(B)$. Hence, $\text{cl}_\beta(A) \cap B = A \cap B = \emptyset$ and $A \cap \text{cl}_\beta(B) = A \cap B = \emptyset$. Therefore, $(X, \tau, I)$ is not $\beta_I$-connected. \hfill \Box

Theorem 11. Let $(X, \tau, I)$ be an ideal topological space and $Y$ be an open set. If $A$ is a $\beta_I$ subset of $X$, then $A \cap Y$ is $\beta_{I_Y}$-open subset of $Y$.

Proof. If $A$ is a $\beta_I$ subset of $X$, then there exists an open set $U'$ such that $U' - A \subseteq I$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. Let $U = U' \cap Y$. Then

$$U - (A \cap Y) = U \cap (A \cap Y)^C = (U' \cap Y) \cap (A^C \cap Y^C) = (U' \cap Y \cap A^C) \cup (U' \cap Y \cap Y^C) = U' \cap Y \cap A^C = (U' - A) \cap Y \in I_Y.$$  

Moreover,

$$(A \cap Y) - \text{cl}(\text{int}(\text{cl}(U))) = (A \cap Y) - \text{cl}(\text{int}(\text{cl}(U' \cap Y))) = (A \cap Y) - \text{cl}(\text{int}(A')) \cap Y = [A - \text{cl}(\text{int}(A'))] \cap Y \in I_Y.$$

Therefore, $A \cap Y$ is $\beta_{I_Y}$-connected. \hfill \Box

Remark 2. Let $(X, \tau, I)$ be an ideal topological space. If $Y \subseteq X$, then $I_Y = \{A \cap Y : A \in I\}$ is a subset of $I$.

The succeeding Theorems are worth-noting.

Theorem 12. Let $(X, \tau, I)$ be an ideal topological space, $Y$ be an open set, and $A \subseteq Y$. $A$ is a $\beta_{I_Y}$-open subset of $Y$ if and only if it is a $\beta_I$-open subset of $X$.

Proof. Assume that $A$ is a $\beta_{I_Y}$-open set. Then there exists an open set $U$ such that $U - A \in I_Y$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. Let $U' = U \cap Y$. Since $Y$ is open, $U'$ is open. Thus, by Remark 2 $U' - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(U'))) \in I$. This shows that $A$ is $\beta_I$-open.

Conversely, assume that $A$ is a $\beta_I$-open set. Then there exists an open set $U$ such that $U - A \in I$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I$. Note that $Y \cap (U - A) \in I_Y$ and $A - \text{cl}(\text{int}(\text{cl}(U))) \in I_Y$, that is $(Y \cap U) - A \in I_Y$ and $A - \text{cl}(\text{int}(Y \cap U))) \in I_Y$. This shows that $A$ is $\beta_{I_Y}$-open. \hfill \Box

Theorem 13. Let $(X, \tau, I)$ be an ideal topological space, $Y$ be an open set, and $A \subseteq Y$. Then $\text{cl}_{\beta_{I_Y}}(A) = \text{cl}_{\beta_I}(A) \cap Y$. 

Proof. Let \( w \notin \text{cl}_{\beta I}(A) \cap Y \). Then \( w \in X - \text{cl}_{\beta I}(A) \). By Theorem 12, \((X - \text{cl}_{\beta I}(A)) \cap Y\) is a \( \beta_I \)-open subset of \( Y \). Note that \( w \) must be in \((X - \text{cl}_{\beta I}(A)) \). Hence, \( Y - [(X - \text{cl}_{\beta I}(A)) \cap Y]\) is a \( \beta_I \)-closed set in \( Y \), which does not contain \( w \). Thus, \( x \notin \text{cl}_{\beta I}(A) \). Therefore, \( \text{cl}_{\beta I}(A) \subseteq \text{cl}_{\beta I}(A) \cap Y \).

Next, let \( z \notin \text{cl}_{\beta I}(A) \). Then \( z \in X - \text{cl}_{\beta I}(A) \). By Theorem 12, \((Y - \text{cl}_{\beta I}(A)) \) is a \( \beta_I \)-open subset of \( X \). Note that \((Y - \text{cl}_{\beta I}(A)) \) must contain \( w \). Thus, \( X - (Y - \text{cl}_{\beta I}(A)) \) is a \( \beta_I \)-closed set in \( X \), which does not contain \( z \). Thus, \( \text{cl}_{\beta I}(A) = \bigcap\{ F : F \) is a \( \beta_I \)-closed set and \( A \subseteq F \}\) does not contain \( z \), that is \( z \notin \text{cl}_{\beta I}(A) \). Therefore, \( \text{cl}_{\beta I}(A) \supseteq \text{cl}_{\beta I}(A) \cap Y \).

\[\blacksquare\]

**Theorem 14.** Let \((X, \tau, I)\) be an ideal topological space, \( Y \) be an open set, and \( A \) and \( B \) be subsets of \( Y \). Then the following statements are equivalent.

(i) \( A \) and \( B \) are \( \beta_I \)-separated in \( Y \).

(ii) \( A \) and \( B \) are \( \beta_I \)-separated in \( X \).

Proof. (1) \(\Rightarrow\) (2) Assume that (1) holds. If \( A \) and \( B \) are \( \beta_I \)-separated in \( Y \), then by the assumption and by Theorem 13 \( \text{cl}_{\beta I}(A) \cap B = \text{cl}_{\beta I}(A) \cap (B \cap Y) = (\text{cl}_{\beta I}(A) \cap Y) \cap B = \text{cl}_{\beta I}(A) \cap B = \emptyset \) and \( A \cap \text{cl}_{\beta I}(B) = (A \cap Y) \cap \text{cl}_{\beta I}(B) = A \cap (\text{cl}_{\beta I}(B) \cap Y) = A \cap \text{cl}_{\beta I}(B) = \emptyset \).

This shows that \( A \) and \( B \) are \( \beta_I \)-separated.

(2) \(\Rightarrow\) (1) Assume that (2) holds. If \( A \) and \( B \) are \( \beta_I \)-separated in \( X \), then by the assumption and by Theorem 13 \( \emptyset = \text{cl}_{\beta I}(A) \cap B = \text{cl}_{\beta I}(A) \cap (B \cap Y) = (\text{cl}_{\beta I}(A) \cap Y) \cap B = \text{cl}_{\beta I}(A) \cap B \) and \( \emptyset = A \cap \text{cl}_{\beta I}(B) = (A \cap Y) \cap \text{cl}_{\beta I}(B) = A \cap (\text{cl}_{\beta I}(B) \cap Y) = A \cap \text{cl}_{\beta I}(B) = \emptyset \).

This shows that \( A \) and \( B \) are \( \beta_I \)-separated.

\[\blacksquare\]

**Theorem 15.** An ideal topological space \((X, \tau, I)\) is a \( \beta_I \)-connected if and only if it cannot be written as a disjoint union of a non-empty \( \beta \)-open set and a \( \beta_I \)-open set.

Proof. Suppose that \((X, \tau, I)\) is a \( \beta_I \)-connected space and \( X \) can be written as a disjoint union of a non-empty \( \beta \)-open set and a \( \beta_I \)-open set. Let \( A \) be a non-empty \( \beta \)-open set and \( B \) be a \( \beta_I \)-open set with \( X = A \cup B \) and \( A \cap B \neq \emptyset \). If \( X = A \cup B \) and \( A \cap B \neq \emptyset \), then \( A \cap B = B \cap A \) since \( A \cap B = B \cap A \). Thus, \( A \cap B \) is a \( \beta_I \)-closed set and \( B \) is a \( \beta_I \)-open set. This implies that \( A \) is a non-empty \( \beta \)-open set and \( B \) is a \( \beta_I \)-open set. This is a contradiction since \( X \) cannot be written as a disjoint union of a non-empty \( \beta \)-open set and a \( \beta_I \)-open set.

Conversely, assume that \( X \) cannot be written as a disjoint union of a non-empty \( \beta \)-open set and a \( \beta_I \)-open set. If \((X, \tau, I)\) is not \( \beta_I \)-connected, then \( X \) can be written as a disjoint union of two \( \beta_I \)-separated sets, say \( A \) and \( B \), with \( X = A \cup B \). Thus, \( \text{cl}_{\beta I}(A) \cap B = \emptyset \) and \( \text{cl}_{\beta I}(B) \cap A = \emptyset \). That is \( \text{cl}_{\beta I}(A) = B \) and \( \text{cl}_{\beta I}(B) = A \). This implies that \( A \) is a non-empty \( \beta \)-open set and \( B \) is a \( \beta_I \)-open set. This is a contradiction since \( X \) cannot be written as a disjoint union of a non-empty \( \beta \)-open set and a \( \beta_I \)-open set.

\[\blacksquare\]

**Theorem 16.** Let \((X, \tau, I)\) be an ideal topological space and \( A \) be an open set. If \( A \) is \( \beta_I \)-connected, and \( H \) and \( G \) are \( \beta_I \)-separated with \( A \subseteq H \cup G \), then either \( A \subseteq H \) or \( A \subseteq G \).
Theorem 17. Let \((X, \tau, I)\) be an ideal topological space and, \(A\) and \(B\) be \(\beta_I\)-separated subsets of \(X\). If \(C\) and \(D\) are two non-empty subsets of \(X\) such that \(C \subseteq D\) and \(D \subseteq B\), then \(C\) and \(D\) are also \(\beta_I\)-separated.

Proof. Suppose that \(A \cap H \neq \emptyset\) and \(A \cap G \neq \emptyset\). Since \(A \subseteq H \cup G\), \(A = (A \cap H) \cup (A \cap G)\). Since \(H\) and \(G\) are \(\beta_I\)-separated, \(\text{cl}_{\beta_I}(A \cap H) \cap (A \cap G) = \emptyset\) and \((A \cap H) \cap \text{cl}_{\beta_I}(A \cap G) = H \cap \text{cl}_{\beta_I}(G) = \emptyset\). Thus, \([\text{cl}_{\beta_I}(A \cap H) \cap A] \cap (A \cap G) = \emptyset\) and \((A \cap H) \cap [\text{cl}_{\beta_I}(A \cap G) \cap A] = \emptyset\). By Theorem 13, \(\text{cl}_{\beta_I}(A \cap H) \cap (A \cap G) = \emptyset\) and \((A \cap H) \cap \text{cl}_{\beta_I}(A \cap G) = \emptyset\). This implies that \(A\) is not \(\beta_I\)-connected. This is a contradiction. Therefore, either \(A \cap H = \emptyset\) or \(A \cap G = \emptyset\), that is \(H \subseteq A\) or \(G \subseteq A\).

Theorem 19. The continuous image a \(\beta_I\)-connected space is connected.

Theorem 20. Let \((X, \tau, I)\) be an ideal topological space. If the union of two \(\beta_I\)-separated sets is a \(\beta\)-closed set, then one of the sets is \(\beta\)-closed and the other is \(\beta_I\)-closed.

Proof. Let \(A\) and \(B\) be \(\beta_I\)-separated such that \(A \cup B\) is \(\beta\)-closed. If \(A\) and \(B\) is \(\beta_I\)-separated, then \(\text{cl}_{\beta_I}(A) \cap B = \emptyset = A \cap \text{cl}_{\beta_I}(B) = \emptyset\). Moreover, if \(A \cup B\) is \(\beta\)-closed, then \(\text{cl}_{\beta}(A \cup B) = A \cup B\). Thus, \(A \subseteq A \cup B\) implies \(\text{cl}_{\beta_I}(A) \subseteq \text{cl}_{\beta_I}(A \cup B) \subseteq \text{cl}_{\beta}(A \cup B) = A \cup B\). Hence, \(\text{cl}_{\beta_I}(A) \subseteq \text{cl}_{\beta_I}(A) \cap (A \cup B) = \text{cl}_{\beta_I}(A) \cap A \cup \text{cl}_{\beta_I}(A) \cap B = \text{cl}_{\beta_I}(A) \cap A = A\), that is \(A\) is \(\beta_I\)-closed. Similarly, \(B \subseteq A \cup B\) implies \(\text{cl}_{\beta}(B) \subseteq \text{cl}_{\beta}(A \cup B) = A \cup B\). Hence, \(\text{cl}_{\beta}(B) \subseteq \text{cl}_{\beta}(B) \cap (A \cup B) = \text{cl}_{\beta}(B) \cap A \cup \text{cl}_{\beta}(B) \cap B = \text{cl}_{\beta}(B) \cap B = B\), that is \(B\) is \(\beta\)-closed.

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