Abstract. This paper introduces the concept of an operation \( \gamma \) on \( \tau_f \). Using this operation, we define the concept of \( f_\gamma \)-open sets, and study some of their related notions. Also, we introduce the concept of \( f_\gamma \)-\( \alpha \)-closed sets and then study some of its properties. Moreover, we introduce and investigate some types of \( f_\gamma \)-separation axioms and \( f_\gamma \)-\( \beta \)-continuous functions by utilizing the operation \( \gamma \) on \( \tau_f \). Finally, some basic properties of functions with \( f_\beta \)-closed graphs have been obtained.

2010 Mathematics Subject Classifications: 54A05, 54A10, 54C05, 54C10, 54D10

Key Words and Phrases: Fine-open sets, \( f_\gamma \)-open sets, \( f_\gamma \)-\( g \)-closed sets, \( f_\gamma \)-separation axioms, \( f_\gamma \)-\( \beta \)-continuous functions, \( f_\beta \)-closed graphs

1. Introduction

Kasahara [11] introduced the notion of an \( \alpha \) operation on a class \( \tau \) of sets and studied the concept of \( \alpha \)-continuous functions with \( \alpha \)-closed graphs and \( \alpha \)-compact spaces. After this, Jankovic [10] introduced the concept of \( \alpha \)-closure of a set in \( X \) via \( \alpha \)-operation and investigated further characterizations of function with \( \alpha \)-closed graph. Later, Ogata [12] defined and studied the concept of \( \gamma \)-open sets, and applied it to investigate operation-functions and operation-separation axioms. Asaad et al. [7] introduced the notion of \( \gamma \)-extremally disconnected spaces. Asaad et al. [5] studied further characterizations of \( \gamma \)-extremally disconnected spaces and investigated some relations of functions of \( \gamma \)-extremally disconnected spaces. Asaad [4] defined a \( \gamma \) operation on generalized open sets in \( X \) and studied its applications. In 2017-2018, Ahmad and Asaad ([1], [6]) introduced an operation \( \gamma \) on semi generalized open subsets of \( X \) and discussed some types of separation axioms, functions and closed spaces with respect to \( \gamma \). Recently, Asaad and Ameen [8] introduced an operation on \( g_\alpha \)-open sets and studied some of its properties. On
the other hand, Powar and Rajak [13] defined the concept of fine-open sets. They studied fine-irresolute homeomorphism and fine-quotient function.

The aim of this paper is to introduce the concept of an operation $\gamma$ on $\tau_f$ and to define the notion of $f_\gamma$-open sets of $(X, \tau, \tau_f)$ by using the operation $\gamma$ on $\tau_f$. Also, some notions of $f_\gamma$-open sets with their relationships are studied. In Section 4, we introduce the concept of $f_\gamma g$-closed sets and then investigate some of its properties. In Section 5, some types of $f_\gamma$-separation axioms by utilizing the operation $\gamma$ on $\tau_f$ are introduced and investigated. In the last two sections, some basic properties of $f_\gamma\beta$-continuous functions with $f_\beta$-closed graphs have been obtained.

2. Preliminaries

Throughout this paper, the space $(X, \tau)$ (or simply $X$) always mean topological space on which no separation axioms are assumed unless explicitly stated.

Definition 2.1. [13] Let $(X, \tau)$ be a topological space, we define $\tau(A_\alpha) = \tau_\alpha$ (say) = \{ $G_\alpha \cap A_\alpha \neq \emptyset$, for $A_\alpha \in \tau$ and $A_\alpha \neq \emptyset$, for some $\alpha \in J$, where $J$ is the index set}. Now, define $\tau_f = \{ \phi, X \} \cup_\alpha \{ \tau_\alpha \}$. The above collection $\tau_f$ of subsets of $X$ is called the fine collection of subsets of $X$ and $(X, \tau, \tau_f)$ is said to be the fine space $X$ generated by the topology $\tau$ on $X$.

Definition 2.2. [13] A subset $U$ of a fine space $X$ is said to be fine-open in $X$, if $U$ belongs to the collection $\tau_f$. It is clear that every open set of $X$ is fine-open in $X$. The complement of a fine-open set of $X$ is called the fine-closed in $X$.

Remark 2.3. [13] Let $(X, \tau, \tau_f)$ be a fine space. Then the following are holds.

(i) The arbitrary union of any fine-open sets in $X$ is fine-open of $X$.

(ii) The intersection of two fine-open sets need not be fine-open.

Definition 2.4. [13] Let $A$ be the subset of a fine space $X$, the fine interior of $A$ is defined as the union of all fine-open sets contained in $A$. That means, the fine interior of $A$ is the largest fine-open set contained in $A$ and it is denoted by $f_{int}(A)$.

Definition 2.5. [13] Let $A$ be the subset of a fine space $X$, the fine closure of $A$ is defined as the intersection of all fine-closed sets containing the set $A$. That means, the fine closure of $A$ is the smallest fine-closed set containing $A$ and it is denoted by $f_{cl}(A)$.

Definition 2.6. [12] An operation $\gamma$ on the topology $\tau$ on $X$ is a mapping $\gamma: \tau \to P(X)$ such that $U \subseteq \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of $X$ and $\gamma(U)$ denotes the value of $\gamma$ at $U$. A nonempty subset $A$ of a topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-open if for each $x \in A$, there exists an open set $U$ containing $x$ such that $\gamma(U) \subseteq A$. The complement of a $\gamma$-open subset of a space $X$ as $\gamma$-closed. The family of all $\gamma$-open subsets of a space $(X, \tau)$ is denoted by $\tau_\gamma$. 
Definition 2.7. [10] A point $x \in X$ is in the $\gamma$-closure of a set $A \subseteq X$ if $\gamma(U) \cap A \neq \emptyset$ for each open set $U$ containing $x$. The set of all $\gamma$-closure points of $A$ is called $\gamma$-closure of $A$ and is denoted by $\text{Cl}_\gamma(A)$.

Definition 2.8. [12] A subset $A$ of $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be $\gamma$-$g$-closed if $\text{Cl}_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\gamma$-open in $(X, \tau)$.

Definition 2.9. [12] A topological space $(X, \tau)$ with an operation $\gamma$ on $\tau$ is said to be

(i) $\gamma$-$T_0$ if for any two distinct points $x, y \in X$, there exists an open set $U$ such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.

(ii) $\gamma$-$T_1$ if for any two distinct points $x, y \in X$, there exist two open sets $U$ and $V$ containing $x$ and $y$ respectively such that $y \notin \gamma(U)$ and $x \notin \gamma(V)$.

(iii) $\gamma$-$T_2$ if for any two distinct points $x, y \in X$, there exist two open sets $U$ and $V$ containing $x$ and $y$ respectively such that $\gamma(U) \cap \gamma(V) = \emptyset$.

(iv) $\gamma$-$T_2$ if every $\gamma$-$g$-closed set in $X$ is $\gamma$-closed.

3. $f_\gamma$-Open Sets

An operation $\gamma$ on $\tau_f$ is a mapping $\gamma: \tau_f \to P(X)$ such that $U \subseteq \gamma(U)$ for every $U \in \tau_f$, where $P(X)$ is the power set of $X$ and $\gamma(U)$ is the value of $\gamma$ at $U$. From this, we can easily to find $\gamma(X) = X$ for any operation $\gamma: \tau_f \to P(X)$. The operators defined by $\gamma(U) = U$, $\gamma(U) = X$, $\gamma(U) = f_{cl}(U)$ and $\gamma(U) = f_{int}(f_{cl}(U))$ are all examples of the operation $\gamma$.

Definition 3.1. Let $(X, \tau, \tau_f)$ be a fine space and $\gamma: \tau_f \to P(X)$ be an operation on $\tau_f$. A nonempty set $A$ of $X$ is said to be $f_\gamma$-open if for each $x \in A$, there exists a fine-open set $U$ such that $x \in U$ and $\gamma(U) \subseteq A$. The complement of a $f_\gamma$-open set of $X$ is $f_\gamma$-closed. Suppose that the empty set $\phi$ is also $f_\gamma$-open set for any operation $\gamma: \tau_f \to P(X)$. The family of all $f_\gamma$-open subsets of a fine space $(X, \tau, \tau_f)$ is denoted by $\tau_{f\gamma}$.

Theorem 3.2. The union of any collection of $f_\gamma$-open sets in a fine space $(X, \tau, \tau_f)$ is a $f_\gamma$-open set in $(X, \tau, \tau_f)$.

Proof. Let $x \in \bigcup_{\alpha \in \Delta} \{A_\alpha\}$, where $\{A_\alpha\}_{\alpha \in \Delta}$ be a class of $f_\gamma$-open sets in $X$. Then $x \in A_\alpha$ for some $\alpha \in \Delta$. Since $A_\alpha$ is $f_\gamma$-open set in $X$, then there exists a fine-open set $V$ such that $x \in V \subseteq \gamma(V) \subseteq A_\alpha \subseteq \bigcup_{\alpha \in \Delta} \{A_\alpha\}$. Therefore, $\bigcup_{\alpha \in \Delta} \{A_\alpha\}$ is a $f_\gamma$-open set in $X$.

Example 3.3. The intersection of any two $f_\gamma$-open sets in $(X, \tau, \tau_f)$ is generally not a $f_\gamma$-open sets. To see this, let $X = \{a, b, c\}$ and $\tau = P(X) = \tau_f$. Let $\gamma: \tau_f \to P(X)$ be an
operation on $\tau_f$ defined as follows:

For every $A \in \tau_f$

$$\gamma(A) = \begin{cases} 
A & \text{if } A \neq \{c\} \\
\{b,c\} & \text{if } A = \{c\}
\end{cases}$$

Thus, $\tau_{f\gamma} = P(X) \setminus \{c\}$. Then $\{a,c\} \in \tau_{f\gamma}$ and $\{b,c\} \in \tau_{f\gamma}$, but $\{a,c\} \cap \{b,c\} = \{c\} \notin \tau_{f\gamma}$. Therefore, $\tau_{f\gamma}$ does not form a topology on $X$.

It is clear from Definition 3.1 that every $f_{\gamma}$-open set is fine-open in $(X, \tau, \tau_f)$ (That is, $\tau_{f\gamma} \subseteq \tau_f$). But the converse need not be true as shown by the following example.

Example 3.4. In Example 3.3, the set $\{c\}$ is fine-open, but it is not $f_{\gamma}$-open.

Definition 3.5. A fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be $f_{\gamma}$-regular if for each $x \in X$ and for each fine-open set $U$ containing $x$, there exists a fine-open set $W$ such that $x \in W$ and $\gamma(W) \subseteq U$.

Theorem 3.6. Let $(X, \tau, \tau_f)$ be a fine space and $\gamma : \tau_f \to P(X)$ be an operation on $\tau_f$. Then the following conditions are equivalent:

(i) $\tau_f = \tau_{f\gamma}$.

(ii) $(X, \tau, \tau_f)$ is a $f_{\gamma}$-regular space.

(iii) For every $x \in X$ and for every fine-open set $U$ of $(X, \tau, \tau_f)$ containing $x$, there exists a $f_{\gamma}$-open set $W$ of $(X, \tau, \tau_f)$ containing $x$ such that $W \subseteq U$.

Proof. (1) $\Rightarrow$ (2) Let $x \in X$ and $U$ be a fine-open set in $X$ such that $x \in U$. It follows from assumption that $U$ is a $f_{\gamma}$-open set. This implies that there exists a fine-open set $W$ such that $x \in W$ and $\gamma(W) \subseteq U$. Therefore, the fine space $(X, \tau, \tau_f)$ is $f_{\gamma}$-regular.

(2) $\Rightarrow$ (3) Let $x \in X$ and $U$ be a fine-open set in $(X, \tau, \tau_f)$ containing $x$. Then by (2), there is a fine-open set $W$ such that $x \in W \subseteq \gamma(W) \subseteq U$. Again, by using (2) for the set $W$, it is shown that $W$ is $f_{\gamma}$-open. Hence $W$ is a $f_{\gamma}$-open set containing $x$ such that $W \subseteq U$.

(3) $\Rightarrow$ (1) By applying the part (3) and Theorem 3.2, it follows that every fine-open set of $X$ is $f_{\gamma}$-open in $X$. That is, $\tau_f \subseteq \tau_{f\gamma}$. But in general, we have $\tau_{f\gamma} \subseteq \tau_f$. Therefore, $\tau_f = \tau_{f\gamma}$.

Definition 3.7. Let $(X, \tau, \tau_f)$ be any fine space. An operation $\gamma$ on $\tau_f$ is said to be

(i) fine-open if for each $x \in X$ and for every fine-open set $U$ containing $x$, there exists a $f_{\gamma}$-open set $W$ containing $x$ such that $W \subseteq \gamma(U)$.

(ii) fine-regular if for each $x \in X$ and for every pair of fine-open sets $U_1$ and $U_2$ such that both containing $x$, there exists a fine-open set $W$ containing $x$ such that $\gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2)$.
Lemma 3.8. Let a mapping \( \gamma \) be fine-regular operation on \( \tau_f \). If \( A \) and \( B \) are \( f_\gamma \)-open sets in a fine space \((X, \tau, \tau_f)\), then \( A \cap B \) is also \( f_\gamma \)-open set in \((X, \tau, \tau_f)\).

Proof. Suppose \( x \in A \cap B \) for any \( f_\gamma \)-open sets \( A \) and \( B \) in \((X, \tau, \tau_f)\) both containing \( x \). Then there exist fine-open sets \( U_1 \) and \( U_2 \) such that \( x \in U_1 \subseteq A \) and \( x \in U_2 \subseteq B \). Since \( \gamma \) is a fine-regular operation on \( \tau_f \), then there exists a fine-open set \( W \) containing \( x \) such that \( \gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \subseteq A \cap B \). Therefore, \( A \cap B \) is \( f_\gamma \)-open set in \((X, \tau, \tau_f)\).

Remark 3.9. By applying Lemma 3.8, it is easy to show that \( \tau_f, \gamma \) forms a topology on \( X \) for any fine-regular operation \( \gamma \) on \( \tau_f \).

Definition 3.10. Let \( A \) be any subset of a fine space \((X, \tau, \tau_f)\) and \( \gamma \) be an operation on \( \tau_f \). The point \( x \in X \) is said to be \( f_\gamma \)-closed of \( A \) if \( \gamma(U) \cap A \neq \emptyset \) for each \( U \in \tau_f \) such that \( x \in U \). We denote \( f\text{cl}_\gamma(A) \) by the \( f_\gamma \)-closure of \( A \) which is the set of all \( f_\gamma \)-closure points of \( A \).

Definition 3.11. Let \( A \) be any subset of a fine space \((X, \tau, \tau_f)\) and \( \gamma \) be an operation on \( \tau_f \). We define \( \tau_f, \gamma \text{-cl}(A) \) as the intersection of all \( f_\gamma \)-closed sets of \( X \) containing \( A \).

\[ \text{i.e. } \tau_f, \gamma \text{-cl}(A) = \bigcap \{ F : A \subseteq F, X \setminus F \in \tau_f, \gamma \} \]

Theorem 3.12. Let \( A \) be any subset of a fine space \((X, \tau, \tau_f)\) and \( \gamma \) be an operation on \( \tau_f \). Then \( x \in \tau_f, \gamma \text{-cl}(A) \) if and only if \( A \cap U \neq \emptyset \) for every \( f_\gamma \)-open set \( U \) of \( X \) containing \( x \).

Proof. Let \( x \in \tau_f, \gamma \text{-cl}(A) \) and let \( A \cap U = \emptyset \) for some \( f_\gamma \)-open set \( U \) of \( X \) containing \( x \). Then \( A \subseteq X \setminus U \) and \( X \setminus U \) is \( f_\gamma \)-closed set in \( X \). So \( \tau_f, \gamma \text{-cl}(A) \subseteq X \setminus U \). Thus, \( x \in X \setminus U \). This is a contradiction. Hence \( A \cap U \neq \emptyset \) for every \( f_\gamma \)-open set \( U \) of \( X \) containing \( x \).

Conversely, suppose that \( x \notin \tau_f, \gamma \text{-cl}(A) \). So there exists a \( f_\gamma \)-closed set \( F \) such that \( A \subseteq F \) and \( x \notin F \). Then \( X \setminus F \) is a \( f_\gamma \)-open set such that \( x \in X \setminus F \) and \( A \cap (X \setminus F) = \emptyset \). Contradiction of hypothesis. Therefore, \( x \in \tau_f, \gamma \text{-cl}(A) \).

Lemma 3.13. The following statements are true for any subsets \( A \) and \( B \) of a fine space \((X, \tau, \tau_f)\) with an operation \( \gamma \) on \( \tau_f \).

(i) \( f\text{cl}_\gamma(A) \) is fine-closed set in \( X \) and \( \tau_f, \gamma \text{-cl}(A) \) is \( f_\gamma \)-closed set in \( X \).

(ii) \( A \subseteq f\text{cl}_\gamma(A) \subseteq \tau_f, \gamma \text{-cl}(A) \).

(iii) \( \tau_f, \gamma \text{-cl}(\emptyset) = f\text{cl}_\gamma(\emptyset) = \emptyset \) and \( \tau_f, \gamma \text{-cl}(X) = f\text{cl}_\gamma(X) = X \).

(iv) (a) \( A \) is \( f_\gamma \)-closed if and only if \( \tau_f, \gamma \text{-cl}(A) = A \) and,

(b) \( A \) is \( f_\gamma \)-closed if and only if \( f\text{cl}_\gamma(A) = A \).

(v) If \( A \subseteq B \), then \( \tau_f, \gamma \text{-cl}(A) \subseteq \tau_f, \gamma \text{-cl}(B) \) and \( f\text{cl}_\gamma(A) \subseteq f\text{cl}_\gamma(B) \).

(vi) (a) \( \tau_f, \gamma \text{-cl}(A \cap B) \subseteq \tau_f, \gamma \text{-cl}(A) \cap \tau_f, \gamma \text{-cl}(B) \) and,
(b) \( fcl_\gamma(A \cap B) \subseteq fcl_\gamma(A) \cap fcl_\gamma(B) \).

(vii) (a) \( \tau_{f_\gamma}\text{-cl}(A) \cup \tau_{f_\gamma}\text{-cl}(B) \subseteq \tau_{f_\gamma}\text{-cl}(A \cup B) \) and,
(b) \( fcl_\gamma(A) \cup fcl_\gamma(B) \subseteq fcl_\gamma(A \cup B) \).

(viii) \( \tau_{f_\gamma}\text{-cl}(\tau_{f_\gamma}\text{-cl}(A)) = \tau_{f_\gamma}\text{-cl}(A) \).

Proof. Straightforward.

Theorem 3.14. For any subsets \( A, B \) of a fine space \((X, \tau, \tau_f)\). If \( \gamma \) is a fine-regular operation on \( \tau_f \), then

(i) \( \tau_{f_\gamma}\text{-cl}(A) \cap \tau_{f_\gamma}\text{-cl}(B) = \tau_{f_\gamma}\text{-cl}(A \cap B) \).

(ii) \( fcl_\gamma(A) \cup fcl_\gamma(B) = fcl_\gamma(A \cup B) \).

Proof. (1) It is enough to prove that \( \tau_{f_\gamma}\text{-cl}(A \cup B) \subseteq \tau_{f_\gamma}\text{-cl}(A) \cup \tau_{f_\gamma}\text{-cl}(B) \) since the other part follows directly from Lemma 3.13 (7). Let \( x \notin \tau_{f_\gamma}\text{-cl}(A) \cup \tau_{f_\gamma}\text{-cl}(B) \). Then by using Theorem 3.12, there exist two \( f_{\gamma}\text{-open} \) sets \( U \) and \( V \) containing \( x \) such that \( A \cap U = \phi \) and \( B \cap V = \phi \). Since \( \gamma \) is a fine-regular operation on \( \tau_f \), then by Lemma 3.8, \( U \cap V \) is \( f_{\gamma}\text{-open} \) in \( X \) such that

\[
(U \cap V) \cap (A \cup B) = \phi.
\]

Therefore, we have \( x \notin \tau_{f_\gamma}\text{-cl}(A \cup B) \) and hence

\[
\tau_{f_\gamma}\text{-cl}(A \cup B) \subseteq \tau_{f_\gamma}\text{-cl}(A) \cup \tau_{f_\gamma}\text{-cl}(B).
\]

(2) Let \( x \notin fcl_\gamma(A) \cup fcl_\gamma(B) \). Then there exist fine-open sets \( U_1 \) and \( U_2 \) such that \( x \in U_1, x \in U_2 \), \( A \cap \gamma(U_1) = \phi \) and \( A \cap \gamma(U_2) = \phi \). Since \( \gamma \) is a fine-regular operation on \( \tau_f \), then there exists a fine-open set \( W \) containing \( x \) such that \( \gamma(W) \subseteq \gamma(U_1) \cap \gamma(U_2) \). Thus, we have

\[
(A \cup B) \cap \gamma(W) \subseteq (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)).
\]

This implies that \( (A \cup B) \cap \gamma(W) = \phi \) since \( (A \cup B) \cap (\gamma(U_1) \cap \gamma(U_2)) = \phi \). This means that \( x \notin fcl_\gamma(A \cup B) \) and hence \( fcl_\gamma(A \cup B) \subseteq fcl_\gamma(A) \cup fcl_\gamma(B) \). Using Lemma 3.13 (7), we have the equality.

Theorem 3.15. Let \( A \) be any subset of a fine space \((X, \tau, \tau_f)\). If \( \gamma \) is a fine-regular operation on \( \tau_f \), then \( fcl_\gamma(A) = \tau_{f_\gamma}\text{-cl}(A) \), \( fcl_\gamma(fcl_\gamma(A)) = fcl_\gamma(A) \) and \( fcl_\gamma(A) \) is \( f_{\gamma}\text{-closed} \) set in \( X \).

Proof. By Lemma 3.13 (2), we have \( fcl_\gamma(A) \subseteq \tau_{f_\gamma}\text{-cl}(A) \). Now, we need to show that \( \tau_{f_\gamma}\text{-cl}(A) \subseteq fcl_\gamma(A) \). Let \( x \notin fcl_\gamma(A) \), then there exists a fine-open set \( U \) containing \( x \) such that \( A \cap \gamma(U) = \phi \). Since \( \gamma \) is a fine-open on \( \tau_f \), then there exists a \( f_{\gamma}\text{-open} \) set \( W \) containing \( x \) such that \( W \subseteq \gamma(U) \). So \( A \cap W = \phi \) and hence by Theorem 3.12, \( x \notin \tau_{f_\gamma}\text{-cl}(A) \). Therefore, \( \tau_{f_\gamma}\text{-cl}(A) \subseteq fcl_\gamma(A) \). Hence \( fcl_\gamma(A) = \tau_{f_\gamma}\text{-cl}(A) \). Moreover, using the above result and by Lemma 3.13 (8), we get \( fcl_\gamma(fcl_\gamma(A)) = fcl_\gamma(A) \) and by Lemma 3.13 (4b), we obtain \( fcl_\gamma(A) \) is \( f_{\gamma}\text{-closed} \) set in \( X \).
Theorem 3.16. Let $A$ be any subset of a fine space $(X, \tau_f)$ and $\gamma$ be an operation on $\tau_f$. Then the following statements are equivalent:

(i) $A$ is $f_\gamma$-open set.

(ii) $fcl_\gamma(X \setminus A) = X \setminus A$.

(iii) $\tau_f \gamma$-cl$(X \setminus A) = X \setminus A$.

(iv) $X \setminus A$ is $f_\gamma$-closed set.

Proof. Clear.

Lemma 3.17. Let $(X, \tau, \tau_f)$ be a fine space and $\gamma$ be a fine-regular operation on $\tau_f$. Then $\tau_f \gamma$-cl$(A) \cap U \subseteq \tau_f \gamma$-cl$(A \cap U)$ holds for every $f_\gamma$-open set $U$ and every subset $A$ of $X$.

Proof. Suppose that $x \in \tau_f \gamma$-cl$(A) \cap U$ for every $f_\gamma$-open set $U$, then $x \in \tau_f \gamma$-cl$(A)$ and $x \in U$. Let $V$ be any $f_\gamma$-open set of $X$ containing $x$. Since $\gamma$ is fine-regular on $\tau_f$. So by Lemma 3.8, $U \cap V$ is $f_\gamma$-open set containing $x$. Since $x \in \tau_f \gamma$-cl$(A)$, then by Theorem 3.12, we have $A \cap (U \cap V) \neq \phi$. This means that $(A \cap U) \cap V \neq \phi$. Therefore, again by Theorem 3.12, we obtain that $x \in \tau_f \gamma$-cl$(A \cap U)$. Thus, $\tau_f \gamma$-cl$(A) \cap U \subseteq \tau_f \gamma$-cl$(A \cap U)$.

4. $f_\gamma g$-Closed Sets

Definition 4.1. A subset $A$ of a fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be $f_\gamma$-generalized closed (briefly $f_\gamma g$-closed) if $fcl_\gamma(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $f_\gamma$-open set in $X$.

Lemma 4.2. Let $(X, \tau, \tau_f)$ be a fine space and $\gamma$ be an operation on $\tau_f$. A set $A$ in $(X, \tau, \tau_f)$ is $f_\gamma g$-closed if and only if $A \cap \tau_f \gamma$-cl$(\{x\}) \neq \phi$ for every $x \in fcl_\gamma(A)$.

Proof. Suppose $A$ is $f_\gamma g$-closed set in $X$ and suppose (if possible) that there exists an element $x \in fcl_\gamma(A)$ such that $A \cap \tau_f \gamma$-cl$(\{x\}) = \phi$. This follows that $A \subseteq X \setminus \tau_f \gamma$-cl$(\{x\})$. Since $\tau_f \gamma$-cl$(\{x\})$ is $f_\gamma$-closed implies $X \setminus \tau_f \gamma$-cl$(\{x\})$ is $f_\gamma$-open and $A$ is $f_\gamma g$-closed set in $X$. Then, we have that $fcl_\gamma(A) \subseteq X \setminus \tau_f \gamma$-cl$(\{x\})$. This means that $x \notin fcl_\gamma(A)$. This is a contradiction. Hence $A \cap \tau_f \gamma$-cl$(\{x\}) \neq \phi$.

Conversely, let $U \in \tau_f$, such that $A \subseteq U$. To show that $fcl_\gamma(A) \subseteq U$. Let $x \in fcl_\gamma(A)$. Then by hypothesis, $A \cap \tau_f \gamma$-cl$(\{x\}) \neq \phi$. So there exists an element $y \in A \cap \tau_f \gamma$-cl$(\{x\})$. Thus $y \in A \subseteq U$ and $y \in \tau_f \gamma$-cl$(\{x\})$. By Theorem 3.12, $\{x\} \cap U \neq \phi$. Hence $x \in U$ and so $fcl_\gamma(A) \subseteq U$. Therefore, $A$ is $f_\gamma g$-closed set in $(X, \tau, \tau_f)$.

Theorem 4.3. Let $A$ be a subset of fine space $(X, \tau, \tau_f)$ and $\gamma$ be an operation on $\tau_f$. If $A$ is $f_\gamma g$-closed, then $fcl_\gamma(A) \setminus A$ does not contain any non-empty $f_\gamma$-closed set.
Theorem 4.4. If \( \gamma \colon \tau_f \rightarrow P(X) \) is a fine-open operation, then the converse of the Theorem 4.3 is true.

Proof. Let \( U \) be a non-empty \( f_\gamma \)-closed in \( X \) such that \( F \subseteq fcl_\gamma(A) \setminus A \). Then \( F \subseteq X \setminus A \) implies \( A \subseteq X \setminus F \). Since \( X \setminus F \) is \( f_\gamma \)-open set and \( A \) is \( f_\gamma \)-closed set, then \( fcl_\gamma(A) \subseteq X \setminus F \). That is \( F \subseteq X \setminus fcl_\gamma(A) \). Hence \( F \subseteq X \setminus fcl_\gamma(A) \) such that \( F \setminus A \) is a \( f_\gamma \)-closed set in \( X \). Using the assumption of the converse of the Theorem 4.3, \( fcl_\gamma(A) \subseteq U \). Therefore, \( A \) is \( f_\gamma \)-closed set in \( (X, \tau, \tau_f) \).

Corollary 4.5. Let \( A \) be a \( f_\gamma \)-closed subset of fine space \( (X, \tau, \tau_f) \) and let \( \gamma \) be an operation on \( \tau_f \). Then \( A \) is \( f_\gamma \)-closed if and only if \( fcl_\gamma(A) \setminus A \) is \( f_\gamma \)-closed set.

Proof. Let \( A \) be a \( f_\gamma \)-closed set in \( (X, \tau, \tau_f) \). Then by Lemma 3.13 (4b), \( fcl_\gamma(A) = A \) and hence \( fcl_\gamma(A) \setminus A = \phi \) which is \( f_\gamma \)-closed set.

Conversely, suppose \( fcl_\gamma(A) \setminus A \) is \( f_\gamma \)-closed and \( A \) is \( f_\gamma \)-closed. Then by Theorem 4.3, \( fcl_\gamma(A) \setminus A \) does not contain any non-empty \( f_\gamma \)-closed set and since \( fcl_\gamma(A) \setminus A \) is \( f_\gamma \)-closed subset of itself, then \( fcl_\gamma(A) \setminus A = \phi \) implies \( fcl_\gamma(A) \setminus A = \phi \). Hence \( fcl_\gamma(A) = A \). This follows from Lemma 3.13 (4b) that \( A \) is \( f_\gamma \)-closed set in \( (X, \tau, \tau_f) \).

Theorem 4.6. Let \( (X, \tau) \) be a fine space and \( \gamma \) be an operation on \( \tau_f \). If a subset \( A \) of \( X \) is \( f_\gamma \)-closed and \( f_\gamma \)-open, then \( A \) is \( f_\gamma \)-closed.

Proof. Since \( A \) is \( f_\gamma \)-closed and \( f_\gamma \)-open set in \( X \), then \( fcl_\gamma(A) \subseteq A \) and hence by Lemma 3.13 (4b), \( A \) is \( f_\gamma \)-closed.

Theorem 4.7. In any fine space \( (X, \tau, \tau_f) \) with an operation \( \gamma \) on \( \tau_f \). For an element \( x \in X \), the set \( X \setminus \{x\} \) is \( f_\gamma \)-closed or \( f_\gamma \)-open.

Proof. Suppose that \( X \setminus \{x\} \) is not \( f_\gamma \)-open. Then \( X \) is the only \( f_\gamma \)-open set containing \( X \setminus \{x\} \). This implies that \( fcl_\gamma(X \setminus \{x\}) \subseteq X \). Thus \( X \setminus \{x\} \) is a \( f_\gamma \)-closed set in \( X \).

Corollary 4.8. In any fine space \( (X, \tau, \tau_f) \) with an operation \( \gamma \) on \( \tau_f \). For an element \( x \in X \), either the set \( \{x\} \) is \( f_\gamma \)-closed or the set \( X \setminus \{x\} \) is \( f_\gamma \)-closed.

Proof. Suppose \( \{x\} \) is not \( f_\gamma \)-closed, then \( X \setminus \{x\} \) is not \( f_\gamma \)-open. Hence by Theorem 4.7, \( X \setminus \{x\} \) is \( f_\gamma \)-closed set in \( X \).

Definition 4.9. Let \( A \) be any subset of a fine space \( (X, \tau, \tau_f) \) and \( \gamma \) be an operation on \( \tau_f \). Then the \( \tau_f \)-kernel of \( A \) is denoted by \( \tau_f \)-ker(\( A \)) and is defined as follows:
In other words, $\tau_{f_\gamma}\ker(A)$ is the intersection of all $f_\gamma$-open sets of $(X, \tau, \tau_f)$ containing $A$.

**Theorem 4.10.** Let $A \subseteq (X, \tau, \tau_f)$ and $\gamma$ be an operation on $\tau_f$. Then $A$ is $f_\gamma g$-closed if and only if $fcl_\gamma(A) \subseteq \tau_{f_\gamma}\ker(A)$.

**Proof.** Suppose that $A$ is $f_\gamma g$-closed. Then $fcl_\gamma(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $f_\gamma$-open. Let $x \in fcl_\gamma(A)$. Then by Lemma 4.2, $A \cap \tau_{f_\gamma}\cl(\{x\}) \neq \emptyset$. So there exists a point $z$ in $X$ such that $z \in A \cap \tau_{f_\gamma}\cl(\{x\})$ implies that $z \in A \subseteq U$ and $z \in \tau_{f_\gamma}\cl(\{x\})$. By Theorem 3.12, $\{x\} \cap U \neq \emptyset$. Hence we show that $x \in \tau_{f_\gamma}\ker(A)$. Therefore, $fcl_\gamma(A) \subseteq \tau_{f_\gamma}\ker(A)$.

Conversely, let $fcl_\gamma(A) \subseteq \tau_{f_\gamma}\ker(A)$. Let $U$ be any $f_\gamma$-open set containing $A$. Let $x$ be a point in $X$ such that $x \in fcl_\gamma(A)$. Then $x \in \tau_{f_\gamma}\ker(A)$. Namely, we have $x \in U$, because $A \subseteq U$ and $U \in \tau_{f_\gamma}\gamma(A)$. That is $fcl_\gamma(A) \subseteq \tau_{f_\gamma}\ker(A) \subseteq U$. Therefore, $A$ is $f_\gamma g$-closed set in $X$.

5. On $f_\gamma$-Separation Axioms

**Definition 5.1.** A fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be
(i) $f_\gamma T_0$ if for any two distinct points $x, y$ in $X$, there exists a fine-open set $U$ such that $x \in U$ and $y \notin \gamma(U)$ or $y \in U$ and $x \notin \gamma(U)$.
(ii) $f_\gamma T_0^*$ if for each pair of distinct points $x, y$ in $X$, there exists a $f_\gamma$-open set $U$ containing one of the points but not the other.

**Definition 5.2.** A fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be
(i) $f_\gamma T_1$ if for any two distinct points $x, y$ in $X$, there exist two fine-open sets $U$ and $V$ such that $x \in U$ and $y \notin \gamma(U)$, $y \in V$ and $x \notin \gamma(V)$.
(ii) $f_\gamma T_1^*$ if for each pair of distinct points $x, y$ in $X$, there exist two $f_\gamma$-open sets $U$ and $V$ such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

**Definition 5.3.** A fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be
(i) $f_\gamma T_2$ if for any two distinct points $x, y$ in $X$, there exist two fine-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $\gamma(U) \cap \gamma(V) = \emptyset$.
(ii) $f_\gamma T_2^*$ if for each pair of distinct points $x, y$ in $X$, there exist $f_\gamma$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Definition 5.4.** A fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$ is said to be $f_\gamma T_1$ if every $f_\gamma g$-closed set in $X$ is $f_\gamma T_1$.

**Theorem 5.5.** For any fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$. Then $(X, \tau, \tau_f)$ is $f_\gamma T_2$ if and only if for each point $x \in X$, the set $\{x\}$ is $f_\gamma$-closed or $f_\gamma$-open.

**Proof.** Let $X$ be a $f_\gamma T_2$ space and let $\{x\}$ is not $f_\gamma$-closed set in $(X, \tau, \tau_f)$. By Corollary 4.8, $X \setminus \{x\}$ is $f_\gamma g$-closed. Since $(X, \tau, \tau_f)$ is $f_\gamma T_2$, then $X \setminus \{x\}$ is $f_\gamma$-closed set which means that $\{x\}$ is $f_\gamma$-open set in $X$. 
Corollary 5.7. Suppose that $f$ is a fine-open operation on $(X, \tau, \tau_f)$. We have to show that $F$ is $f_{\gamma}$-closed (that is $f_{\gamma}(F) = F$ (by Lemma 3.13 (4b))). It is sufficient to show that $f_{\gamma}\mathcal{L}(F) \subseteq F$. Let $x \in f_{\gamma}\mathcal{L}(F)$. By hypothesis $\{x\}$ is $f_{\gamma}$-closed or $f_{\gamma}$-open for each $x \in X$. So we have two cases: 

Case (1): If $\{x\}$ is $f_{\gamma}$-closed. Suppose $x \notin F$, then $x \in f_{\gamma}\mathcal{L}(F) \setminus F$ contains a non-empty $f_{\gamma}$-closed set $\{x\}$. A contradiction since $F$ is $f_{\gamma}\mathcal{L}$-closed and set and according to the Theorem 4.3. Hence $x \in F$. This follows that $f_{\gamma}\mathcal{L}(F) \subseteq F$ and hence $f_{\gamma}\mathcal{L}(F) = F$. This means from by Lemma 3.13 (4b) that $F$ is $f_{\gamma}$-closed set in $(X, \tau, \tau_f)$. Thus $(X, \tau, \tau_f)$ is $f_{\gamma}\mathcal{T}_F$ space.

Case (2): If $\{x\}$ is $f_{\gamma}$-open set. Then by Theorem 3.12, $F \cap \{x\} \neq \phi$ which implies that $x \in F$. So $f_{\gamma}\mathcal{L}(F) \subseteq F$. Thus by Lemma 3.13 (4b), $F$ is $f_{\gamma}$-closed. Therefore, $(X, \tau, \tau_f)$ is $f_{\gamma}\mathcal{T}_F$ space.

Theorem 5.6. For any fine space $(X, \tau, \tau_f)$ with an operation $\gamma$ on $\tau_f$, we have

(i) Let $\gamma$ be a fine-open operation on $\tau_f$. Then a space $X$ is a $f_{\gamma}\mathcal{T}_0$ space if and only if $f_{\gamma}\mathcal{L}(\{x\}) \neq f_{\gamma}\mathcal{L}(\{y\})$, for every pair $x, y$ of $X$ with $x \neq y$.

(ii) A space $X$ is $f_{\gamma}\mathcal{T}_0^\sigma$ if and only if $\tau_{f_{\gamma}}\mathcal{L}(\{x\}) \neq \tau_{f_{\gamma}}\mathcal{L}(\{y\})$, for every pair of distinct points $x, y$ of $X$.

Proof. (1) Let $x, y$ be any two distinct points of a $f_{\gamma}\mathcal{T}_0$ space $(X, \tau, \tau_f)$. Then by definition, we assume that there exists a $f_{\gamma}$-open set $U$ such that $x \in U$ and $y \notin \gamma(U)$. Since $\gamma$ is a fine-open operation on $\tau_f$, then there exists a $f_{\gamma}$-open set $W$ such that $x \in W$ and $W \subseteq \gamma(U)$. Hence $y \in X \setminus \gamma(U) \subseteq X \setminus W$. Since $X \setminus W$ is a $f_{\gamma}$-closed in $(X, \tau, \tau_f)$. Then we obtain that $f_{\gamma}\mathcal{L}(\{y\}) \subseteq X \setminus W$ and therefore $f_{\gamma}\mathcal{L}(\{y\}) \neq f_{\gamma}\mathcal{L}(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, we have $f_{\gamma}\mathcal{L}(\{x\}) \neq f_{\gamma}\mathcal{L}(\{y\})$. Now, we assume that there exists $z \in X$ such that $z \in f_{\gamma}\mathcal{L}(\{x\})$, but $z \notin f_{\gamma}\mathcal{L}(\{y\})$. If $x \in f_{\gamma}\mathcal{L}(\{y\})$, then $\{x\} \subseteq f_{\gamma}\mathcal{L}(\{y\})$, which implies that $f_{\gamma}\mathcal{L}(\{x\}) \subseteq f_{\gamma}\mathcal{L}(\{y\})$ (by Lemma 3.13 (5)). This implies that $z \in f_{\gamma}\mathcal{L}(\{y\})$. This contradiction shows that $x \notin f_{\gamma}\mathcal{L}(\{y\})$. This means that by Definition 3.10, there exists a fine-open set $U$ such that $x \in U$ and $\gamma(U) \cap \{y\} = \phi$. Thus, we have that $x \in U$ and $y \notin \gamma(U)$. It gives that the fine space $(X, \tau, \tau_f)$ is $f_{\gamma}\mathcal{T}_0$.

(2) Let $X$ be a $f_{\gamma}\mathcal{T}_0^\sigma$ space and $x, y$ be any two distinct points of $X$. Then there exists a $f_{\gamma}$-open set $G$ containing $x$ or $y$ (say $x$, but not $y$). So $X \setminus G$ is a $f_{\gamma}$-closed set, which does not contain $x$, but contains $y$. Since $\tau_{f_{\gamma}}\mathcal{L}(\{y\})$ is the smallest $f_{\gamma}$-closed set containing $y$, $\tau_{f_{\gamma}}\mathcal{L}(\{y\}) \subseteq X \setminus G$, and so $x \notin \tau_{f_{\gamma}}\mathcal{L}(\{y\})$. Therefore, $\tau_{f_{\gamma}}\mathcal{L}(\{x\}) \neq \tau_{f_{\gamma}}\mathcal{L}(\{y\})$.

Conversely, suppose for any $x, y \in X$ with $x \neq y$, $\tau_{f_{\gamma}}\mathcal{L}(\{x\}) \neq \tau_{f_{\gamma}}\mathcal{L}(\{y\})$. Now, let $z \in X$ such that $z \in \tau_{f_{\gamma}}\mathcal{L}(\{x\})$, but $z \notin \tau_{f_{\gamma}}\mathcal{L}(\{y\})$. Now, we claim that $x \in \tau_{f_{\gamma}}\mathcal{L}(\{y\})$.

For, if $x \in \tau_{f_{\gamma}}\mathcal{L}(\{y\})$, then $\{x\} \subseteq \tau_{f_{\gamma}}\mathcal{L}(\{y\})$, which implies that $\tau_{f_{\gamma}}\mathcal{L}(\{x\}) \subseteq \tau_{f_{\gamma}}\mathcal{L}(\{y\})$. This is a contradiction to the fact that $z \notin \tau_{f_{\gamma}}\mathcal{L}(\{y\})$. Hence $x$ belongs to the $f_{\gamma}$-open set $X \setminus \tau_{f_{\gamma}}\mathcal{L}(\{y\})$ to which $y$ does not belong. It gives that $X$ is $f_{\gamma}\mathcal{T}_0^\sigma$ space.

Corollary 5.7. Suppose that $\gamma$ is a fine-open operation on $\tau_f$. A fine space $(X, \tau, \tau_f)$ is $f_{\gamma}\mathcal{T}_0$ if and only if $(X, \tau, \tau_f)$ is $f_{\gamma}\mathcal{T}_0^\sigma$. 

Proof. This follows from Theorem 5.6 and the fact that \( f\text{cl}_\gamma(A) = \tau_{f\gamma}\text{cl}(A) \) for any \( A \subseteq X \) holds under the assumption that \( \gamma \) is a fine-open operation on \( \tau_f \) (see Theorem 3.15).

**Theorem 5.8.** For a fine space \((X, \tau, \tau_f)\) with an operation \( \gamma \) on \( \tau_f \). Then the following statements are true:

(i) \((X, \tau, \tau_f)\) is \(f_{\gamma}-T_1\).

(ii) For every point \(x \in X\), the set \(\{x\}\) is \(f_{\gamma}\)-closed.

(iii) \((X, \tau, \tau_f)\) is \(f_{\gamma}-T_1^*\).

**Proof.** (1) \(\Rightarrow\) (2) Let \(x\) be a point of an \(f_{\gamma}-T_1\) space \((X, \tau, \tau_f)\). Then for any point \(y \in X\) such that \(x \neq y\), there exists a fine-open set \(V_y\) such that \(y \in V_y\) but \(x \notin \gamma(V_y)\). Thus, \(y \in \gamma(V_y) \subseteq X \backslash \{x\}\). This implies that \(X \setminus \{x\} = \cup \{\gamma(V_y) : y \in X \setminus \{x\}\}\). It is shown that \(X \setminus \{x\}\) is \(f_{\gamma}\)-open set in \((X, \tau, \tau_f)\). Hence \(\{x\}\) is \(f_{\gamma}\)-closed set in \((X, \tau, \tau_f)\).

(2) \(\Rightarrow\) (3) Suppose every singleton set in \(X\) is \(f_{\gamma}\)-closed. Let \(x, y \in X\) such that \(x \neq y\). This implies that \(x \in X \setminus \{y\}\). By hypothesis, we get \(X \setminus \{y\}\) is a \(f_{\gamma}\)-open set contains \(x\) but not \(y\). Similarly \(X \setminus \{x\}\) is a \(f_{\gamma}\)-open set contains \(y\) but not \(x\). Therefore, \(X\) is \(f_{\gamma}-T_1^*\) space.

(3) \(\Rightarrow\) (1) It is shown that if \(x \in U\), where \(U \subseteq \tau_f\), then there exist a fine-open set \(V\) such that \(x \in V \subseteq \gamma(V) \subseteq U\). Applying the part (3), we obtain \((X, \tau, \tau_f)\) is \(f_{\gamma}-T_1\).

**Theorem 5.9.** For any fine space \((X, \tau, \tau_f)\) and any operation \(\gamma\) on \(\tau_f\), the following properties hold.

(i) Every \(f_{\gamma}-T_2\) space is \(f_{\gamma}-T_1\).

(ii) Every \(f_{\gamma}-T_1\) space is \(f_{\gamma}-T_2^*\).

(iii) Every \(f_{\gamma}-T_2^*\) space is \(f_{\gamma}-T_0^*\).

(iv) Every \(f_{\gamma}-T_n^*\) space is \(f_{\gamma}-T_{n-1}\), where \(n \in \{2, 1\}\).

(v) Every \(f_{\gamma}-T_n^*\) space is \(f_{\gamma}-T_n\), where \(n \in \{2, 0\}\).

**Proof.** Follows directly from their definitions.

**Remark 5.10.** By Theorem 5.9 and Theorem 5.8, we obtain the following diagram of implications. Moreover, the following Examples 5.11, 5.12, 5.13 and 5.14 below show that the reverse implications are not true in general.
Example 5.11. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\tau_f = \tau \cup \{\{a, c\}, \{b, c\}\}$.

(i) Define an operation $\gamma$ on $\tau_f$ as follows: For every $A \in \tau_f$

$$
\gamma(A) = \begin{cases} 
\{a, b\} & \text{if } A = \{b\} \\
\{b, c\} & \text{if } A = \{c\} \text{ or } \{b, c\} \\
X & \text{otherwise}
\end{cases}
$$

Obviously, the space $(X, \tau, \tau_f)$ is $f_\gamma T_0$, but it is not $f_\gamma T_0^*$. Hence the fine space $(X, \tau, \tau_f)$ is not $f_\gamma T_2^*$.

(ii) Let $\gamma: \tau_f \rightarrow P(X)$ be an operation on $\tau_f$ defined as follows:

For every set $A \in \tau_f$

$$
\gamma(B) = \begin{cases} 
B & \text{if } B = \{a\} \text{ or } \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\} \\
X & \text{otherwise}
\end{cases}
$$

Thus, $\tau_f(\gamma) = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Clearly, the space $(X, \tau, \tau_f)$ is $f_\gamma T_2^*$, but it is not $f_\gamma T_1^*$.

Example 5.12. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{b\}\}$. Then $\tau_f = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}\}$. Let $\gamma: \tau_f \rightarrow P(X)$ be an operation on $\tau_f$ defined as follows:

For every set $A \in \tau_f$

$$
\gamma(A) = \begin{cases} 
A & \text{if } A = \{b\} \text{ or } \{a, b\} \\
f(A) & \text{otherwise}
\end{cases}
$$

Thus, $\tau_f(\gamma) = \{\phi, X, \{b\}, \{a, b\}\}$. Then the fine space $(X, \tau, \tau_f)$ is $f_\gamma T_0^*$, but it is not $f_\gamma T_1^*$. Since $\{b, c\}$ is $f_\gamma g$-closed set in $(X, \tau, \tau_f)$, but $\{b, c\}$ is not $f_\gamma$-closed set in $(X, \tau, \tau_f)$. Therefore, $(X, \tau, \tau_f)$ is not a $f_\gamma T_1^*$ space.

Example 5.13. Suppose $X = \{a, b, c\}$ and $\tau = \text{all subsets of } X$. Define an operation $\gamma$ on $\tau_f$ as follows: For every $A \in \tau_f$

$$
\gamma(A) = \begin{cases} 
A & \text{if } A = \{a, b\} \text{ or } \{a, c\} \text{ or } \{b, c\} \\
X & \text{otherwise}
\end{cases}
$$

Therefore, $(X, \tau, \tau_f)$ is $f_\gamma T_1^*$ space, and by Theorem 5.8, it is $f_\gamma T_1$, but $(X, \tau, \tau_f)$ is not $f_\gamma T_2$ and hence it is not $f_\gamma T_2^*$.

Example 5.14. Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}\}$. Then $\tau_f = \tau \cup \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Define an operation $\gamma: \tau_f \rightarrow P(X)$ by $\gamma(A) = A$ for all $A \in \tau_f$. Here, $\tau_f(\gamma) = \tau_f$ and $\tau_f = \tau$.

Then the fine space $(X, \tau, \tau_f)$ is $f_\gamma T_1$, but it is not $\gamma T_i$ for $i = 0, 1, 2$. 
6. \(f_{\gamma\beta}\)-Continuous Functions

Throughout Section 6 and Section 7, let \((X, \tau, \tau_f)\) and \((Y, \sigma, \sigma_f)\) be two fine spaces and let \(\gamma: \tau_f \rightarrow P(X)\) and \(\beta: \sigma_f \rightarrow P(Y)\) be operations on \(\tau_f\) and \(\sigma_f\) respectively. In this section, we introduce a new class of functions called \(f_{\gamma\beta}\)-continuous. Some characterizations and properties of this function are investigated.

**Definition 6.1.** A function \(h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)\) is said to be \(f_{\gamma\beta}\)-continuous if for each \(x \in X\) and each fine-open set \(V\) containing \(h(x)\), there exists a fine-open set \(U\) containing \(x\) such that \(h(\gamma(U)) \subseteq \beta(V)\).

**Theorem 6.2.** Let \(h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)\) be a \(f_{\gamma\beta}\)-continuous function, then,

(i) \(h(fcl_\gamma(A)) \subseteq fcl_\beta(h(A))\), for every \(A \subseteq (X, \tau, \tau_f)\).

(ii) \(h^{-1}(F)\) is \(f_{\gamma}\)-closed set in \((X, \tau, \tau_f)\), for every \(f_{\beta}\)-closed set \(F\) of \((Y, \sigma, \sigma_f)\).

**Proof.** (1) Let \(y \in h(fcl_\gamma(A))\) and \(V\) be any fine-open set containing \(y\). Then by hypothesis, there exists \(x \in X\) and fine-open set \(U\) containing \(x\) such that \(h(x) = y\) and \(h(\gamma(U)) \subseteq \beta(V)\). Since \(x \in fcl_\gamma(A)\), we have \(\gamma(U) \cap A \neq \emptyset\). Hence \(\emptyset \neq h(\gamma(U) \cap A) \subseteq h(\gamma(U)) \cap h(A) \subseteq \beta(V) \cap h(A)\). This implies that \(y \in fcl_\beta(h(A))\). Therefore, \(h(fcl_\gamma(A)) \subseteq fcl_\beta(h(A))\).

(2) Let \(F\) be any \(f_{\beta}\)-closed set of \((Y, \sigma, \sigma_f)\). By using (1), we have \(h(fcl_\gamma(h^{-1}(F))) \subseteq fcl_\beta(F) = F\). Therefore, \(fcl_\gamma(h^{-1}(F)) = h^{-1}(F)\). Hence \(h^{-1}(F)\) is \(f_{\gamma}\)-closed set in \((X, \tau, \tau_f)\).

**Theorem 6.3.** In Theorem 6.2, the properties of \(f_{\gamma\beta}\)-continuity of \(f\), (1) and (2) are equivalent to each other if either the fine space \((Y, \sigma, \sigma_f)\) is \(f_{\beta}\)-regular or the operation \(\beta\) is fine-open.

**Proof.** It follows from the proof of Theorem 6.2 that we know the following implications:

"\(f_{\gamma\beta}\)-continuity of \(h\)" \(\Rightarrow\) (1) \(\Rightarrow\) (2). Thus, when the fine space \((Y, \sigma, \sigma_f)\) is \(f_{\beta}\)-regular, we prove the implication: (2) \(\Rightarrow\) \(f_{\gamma\beta}\)-continuity of \(h\). Let \(x \in X\) and let \(V \in \sigma_f\) such that \(h(x) \in V\). Since \((Y, \sigma, \sigma_f)\) is a \(f_{\beta}\)-regular space, then by Theorem 3.6, \(V \in \sigma_{g\beta}\). By using (2) of Theorem 6.2, \(h^{-1}(V) \in \tau_f\), such that \(x \in h^{-1}(V)\). So there exists a fine-open set \(U\) such that \(x \in U\) and \(\gamma(U) \subseteq h^{-1}(V)\). This implies that \(h(\gamma(U)) \subseteq V \subseteq \beta(V)\). Therefore, \(h\) is \(f_{\gamma\beta}\)-continuous.

Now, when \(\beta\) is a fine-open operation, we show the implication: (2) \(\Rightarrow\) \(f_{\gamma\beta}\)-continuity of \(h\). Let \(x \in X\) and let \(V \in \sigma_f\) such that \(h(x) \in V\). Since \(\beta\) is a fine-open operation, then there exists \(W \in \sigma_{g\beta}\) such that \(h(x) \in W\) and \(W \subseteq \beta(V)\). By using (2) of Theorem 6.2, \(h^{-1}(W) \in \tau_f\), such that \(x \in h^{-1}(W)\). So there exists a fine-open set \(U\) such that \(x \in U\) and \(\gamma(U) \subseteq h^{-1}(W) \subseteq h^{-1}(\beta(V))\). This implies that \(h(\gamma(U)) \subseteq \beta(V)\). Hence \(h\) is \(f_{\gamma\beta}\)-continuous.

**Definition 6.4.** A function \(h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)\) is said to be
Suppose that a function $h: (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is both $f_{\gamma\beta}$-continuous and $f_\beta$-closed, then:

(i) $f_\beta$-closed if the image of each $f_{\gamma}$-closed set of $X$ is $f_\beta$-closed in $Y$.

(ii) $f_\beta$-closed if the image of each fine-closed set of $X$ is $f_\beta$-closed in $Y$.

**Theorem 6.5.** Suppose that a function $h: (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ is both $f_{\gamma\beta}$-continuous and $f_\beta$-closed, then:

(i) For every $f_{\gamma}g$-closed set $A$ of $(X,\tau,\tau_f)$, the image $h(A)$ is $f_{\gamma\beta}g$-closed in $(Y,\sigma,\sigma_f)$.

(ii) For every $f_{\beta}g$-closed set $B$ of $(Y,\sigma,\sigma_f)$, the inverse set $h^{-1}(B)$ is $f_{\gamma}g$-closed in $(X,\tau,\tau_f)$.

**Proof.** (1) Let $G$ be any $f_\beta$-open set in $(Y,\sigma,\sigma_f)$ such that $h(A) \subseteq G$. Since $h$ is $f_{\gamma\beta}$-continuous function, then by using Theorem 6.2 (2), $h^{-1}(G)$ is $f_\gamma$-open in $(X,\tau,\tau_f)$. Since $A$ is $f_{\gamma}g$-closed and $A \subseteq h^{-1}(G)$, we have $fcl_\gamma(A) \subseteq h^{-1}(G)$, and hence $h(fcl_\gamma(A)) \subseteq G$. Thus, by Lemma 3.13 (1), $fcl_\gamma(A)$ is fine-closed set and since $h$ is $f_\beta$-closed, then $h(fcl_\gamma(A))$ is $f_\beta$-closed set in $Y$. Therefore, $fcl_\beta(h(A)) \subseteq fcl_\beta(h(fcl_\gamma(A))) = h(fcl_\gamma(A)) \subseteq G$. This implies that $h(A)$ is $f_{\gamma\beta}g$-closed in $(Y,\sigma,\sigma_f)$.

2) Let $H$ be any $f_\gamma$-open set of a fine space $(X,\tau,\tau_f)$ such that $h^{-1}(B) \subseteq H$. Let $C = fcl_\gamma(h^{-1}(B)) \cap (X \setminus H)$, then by Lemma 3.13 (1), $C$ is fine-closed set in $(X,\tau,\tau_f)$. Since $h$ is $f_\beta$-closed function. Then $h(C)$ is $f_\beta$-closed in $(Y,\sigma,\sigma_f)$. Since $h$ is $f_{\gamma\beta}$-continuous function, then by using Theorem 6.2 (1), we have $h(C) = h(fcl_\gamma(h^{-1}(B))) \cap h(X \setminus H) \subseteq fcl_\beta(B) \cap h(X \setminus H) \subseteq fcl_\beta(B) \cap (Y \setminus B) = fcl_\beta(B) \setminus B$. This implies from Theorem 4.3 that $h(C) = \phi$, and hence $C = \phi$. So $fcl_\gamma(h^{-1}(B)) \subseteq H$. Therefore, $h^{-1}(B)$ is $f_{\gamma}g$-closed in $(X,\tau,\tau_f)$.

**Theorem 6.6.** Let $h: (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ be an injective, $f_{\gamma\beta}$-continuous and $f_\beta$-closed function. If $(Y,\sigma,\sigma_f)$ is $f_\beta$-$T_{1\frac{1}{2}}$, then $(X,\tau,\tau_f)$ is $f_{\gamma}$-$T_{1\frac{1}{2}}$.

**Proof.** Let $G$ be any $f_{\gamma}g$-closed set of $(X,\tau,\tau_f)$. Since $h$ is $f_{\gamma\beta}$-continuous and $f_\beta$-closed function. Then by Theorem 6.5 (1), $h(G)$ is $f_{\gamma\beta}g$-closed in $(Y,\sigma,\sigma_f)$. Since $(Y,\sigma,\sigma_f)$ is $f_{\beta}$-$T_{1\frac{1}{2}}$, then $h(G)$ is $f_\beta$-closed in $Y$. Again, since $h$ is $f_{\gamma\beta}$-continuous, then by Theorem 6.2 (2), $h^{-1}(h(G))$ is $f_\gamma$-closed in $X$. Hence $G$ is $f_\gamma$-closed in $X$ since $h$ is injective. Therefore, $(X,\tau,\tau_f)$ is a $f_\gamma$-$T_{1\frac{1}{2}}$ space.

**Theorem 6.7.** Let a function $h: (X,\tau,\tau_f) \to (Y,\sigma,\sigma_f)$ be surjective, $f_{\gamma\beta}$-continuous and $f_\beta$-closed. If $(X,\tau,\tau_f)$ is $f_{\gamma}$-$T_{1\frac{1}{2}}$, then $(Y,\sigma,\sigma_f)$ is $f_{\beta}$-$T_{1\frac{1}{2}}$.

**Proof.** Let $H$ be a $f_{\gamma\beta}g$-closed set of $(Y,\sigma,\sigma_f)$. Since $h$ is $f_{\gamma\beta}$-continuous and $f_\beta$-closed function. Then by Theorem 6.5 (2), $h^{-1}(H)$ is $f_{\gamma}g$-closed in $(X,\tau,\tau_f)$. Since $(X,\tau,\tau_f)$ is $f_{\gamma}$-$T_{1\frac{1}{2}}$, then we have, $h^{-1}(H)$ is $f_{\gamma}$-closed set in $X$. Again, since $h$ is $f_\beta$-closed function, then $h(h^{-1}(H))$ is $f_\beta$-closed in $Y$. Therefore, $H$ is $f_\beta$-closed in $Y$ since $h$ is surjective. Hence $(Y,\sigma,\sigma_f)$ is $f_{\beta}$-$T_{1\frac{1}{2}}$ space.
Theorem 6.8. If a function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is injective \( f_{\gamma \beta} \)-continuous and the fine space \( (Y, \sigma, \sigma_f) \) is \( f_\beta - T_2 \), then the fine space \( (X, \tau, \tau_f) \) is \( f_\gamma - T_2 \).

Proof. Let \( x_1 \) and \( x_2 \) be any distinct points of a fine space \( (X, \tau, \tau_f) \). Since \( h \) is an injective function and \( (Y, \sigma, \sigma_f) \) is \( f_\beta - T_2 \). Then there exist two fine-open sets \( U_1 \) and \( U_2 \) in \( Y \) such that \( f(x_1) \in U_1 \), \( h(x_2) \in U_2 \) and \( \beta(U_1) \cap \beta(U_2) = \phi \). Since \( h \) is \( f_{\gamma \beta} \)-continuous, there exist fine-open sets \( V_1 \) and \( V_2 \) in \( X \) such that \( x_1 \in V_1 \), \( x_2 \in V_2 \), \( h(\gamma(V_1)) \subseteq \beta(U_1) \) and \( h(\gamma(V_2)) \subseteq \beta(U_2) \). Therefore \( \beta(U_1) \cap \beta(U_2) = \phi \). Hence \( (X, \tau, \tau_f) \) is \( f_\gamma - T_2 \).

Theorem 6.9. If a function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is injective \( f_{\gamma \beta} \)-continuous and the fine space \( (Y, \sigma, \sigma_f) \) is \( f_\beta - T_1 \), then the fine space \( (X, \tau, \tau_f) \) is \( f_\gamma - T_i \) for \( i \in \{0, 1\} \).

Proof. The proof is similar to Theorem 6.8.

Definition 6.10. A function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is said to be \( f_{\gamma \beta} \)-homeomorphism if \( h \) is bijective, \( f_{\gamma \beta} \)-continuous and \( h^{-1} \) is \( f_\beta \)-continuous.

Theorem 6.11. Assume that a function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is \( f_{\gamma \beta} \)-homeomorphism. If \( (X, \tau, \tau_f) \) is \( f_\gamma - T_2 \), then \( (Y, \sigma, \sigma_f) \) is \( f_\beta - T_2 \).

Proof. Let \( \{y\} \) be any singleton set of \( (Y, \sigma, \sigma_f) \). Then there exists an element \( x \) of \( X \) such that \( y = h(x) \). So by hypothesis and Theorem 5.5, we have \( \{x\} \) is \( f_\gamma \)-closed or \( f_\gamma \)-open set in \( X \). By using Theorem 6.2, \( \{y\} \) is \( f_\beta \)-closed or \( f_\beta \)-open set. Hence the fine space by Theorem 5.5, \( (Y, \sigma, \sigma_f) \) is \( f_\beta - T_2 \).

7. Functions with \( f_\beta \)-Closed Graphs

For a function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \), the subset \( \{(x, h(x)) : x \in X\} \) of the product space \( (X \times Y, \tau \times \sigma) \) is called the graph of \( h \) and is denoted by \( G(h) \) [9]. In this section, we further investigate general operator approaches of closed graphs of functions. Let \( \lambda: (\tau \times \sigma)_f \to P(X \times Y) \) be an operation on \( (\tau \times \sigma)_f \).

Definition 7.1. The graph \( G(h) \) of \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) is called \( f_\beta \)-closed if for each \( (x, y) \in (X \times Y) \setminus G(h) \), there exist fine-open sets \( U \subseteq X \) and \( V \subseteq Y \) containing \( x \) and \( y \), respectively, such that \( (U \times \beta(V)) \cap G(h) = \phi \).

The proof of the following lemma follows directly from the above definition.

Lemma 7.2. A function \( h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f) \) has \( f_\beta \)-closed graph if and only if for each \( (x, y) \in (X \times Y) \setminus G(h) \), there exist \( U \in \tau_f \) containing \( x \) and \( V \in \sigma_f \) containing \( y \) such that \( h(U) \cap \beta(V) = \phi \).

Definition 7.3. An operation \( \lambda: (\tau \times \sigma)_f \to P(X \times Y) \) is said to be fine-associated with \( \gamma \) and \( \beta \) if \( \lambda(U \times V) = \gamma(U) \times \beta(V) \) holds for each \( U \in \tau_f \) and \( V \in \sigma_f \).
Definition 7.4. The operation $\lambda: (\tau \times \sigma)f \rightarrow P(X \times Y)$ is said to be fine-regular with respect to $\gamma$ and $\beta$ if for each $(x,y) \in X \times Y$ and each fine-open set $W$ containing $(x,y)$, there exist fine-open sets $U$ in $X$ and $V$ in $Y$ such that $x \in U$, $y \in V$ and $\gamma(U) \times \beta(V) \subseteq \lambda(W)$.

Theorem 7.5. Let $\lambda: (\tau \times \sigma)f \rightarrow P(X \times X)$ be a fine-associated operation with $\gamma$ and $\gamma$. If $h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)$ is a $f_{\gamma\beta}$-continuous function and $(Y, \sigma, \sigma_f)$ is a $f_{\beta}$-$T_2$ space, then the set $A = \{(x,y) \in X \times X : h(x) = h(y)\}$ is a $f_{\lambda\gamma}$-closed set of $(X \times X, \tau \times \tau)$.

Proof. We want to prove that $fcl_\lambda(A) \subseteq A$. Let $(x,y) \in (X \times X) \setminus A$. Since $(Y, \sigma, \sigma_f)$ is $f_{\beta}$-$T_2$, then there exist two fine-open sets $U$ and $V$ in $(Y, \sigma, \sigma_f)$ such that $h(x) \in U$, $h(y) \in V$ and $\beta(U) \cap \beta(V) = \emptyset$. Moreover, for $U$ and $V$ there exist fine-open sets $R$ and $S$ in $(X, \tau, \tau_f)$ such that $x \in R$, $y \in S$ and $h(\gamma(R)) \subseteq \beta(U)$ and $h(\gamma(S)) \subseteq \beta(V)$ since $h$ is $f_{\gamma\beta}$-continuous. Therefore we have $(x,y) \in \gamma(R) \times \gamma(S) = \lambda(R \times S) \cap A = \emptyset$ because $R \times S \subseteq (\tau \times \tau_f)$. This shows that $(x,y) \notin fcl_\lambda(A)$.

Corollary 7.6. Suppose $\lambda: (\tau \times \tau)f \rightarrow P(X \times X)$ is a fine-associated operation with $\gamma$ and $\gamma$, and it is fine-regular with $\gamma$ and $\gamma$. A finite space $(X, \tau, \tau_f)$ is $f_{\gamma}$-$T_2$ if and only if the diagonal set $\Delta = \{(x,x) : x \in X\}$ is $f_{\lambda\gamma}$-closed of $(X \times X, \tau \times \tau)$.

Theorem 7.7. Let $\lambda: (\tau \times \sigma)f \rightarrow P(X \times Y)$ be a fine-associated operation with $\gamma$ and $\gamma$. If $h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)$ is $f_{\gamma\beta}$-continuous and $(Y, \sigma, \sigma_f)$ is $f_{\beta}$-$T_2$, then the graph of $h$, $G(h) = \{(x,h(x)) : x \in X \times Y\}$ is a $f_{\lambda\gamma}$-closed set of $(X \times Y, \tau \times \sigma)$.

Proof. The proof is similar to Theorem 7.5.

Definition 7.8. Let $(X, \tau, \tau_f)$ be a finite space and $\gamma$ be an operation on $\tau_f$. A subset $S$ of $X$ is said to be $f_{\gamma}$-compact if for every fine-open cover $\{U_i, i \in N\}$ of $S$, there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ such that $S \subseteq \gamma(U_1) \cup \gamma(U_2) \cup ... \cup \gamma(U_n)$.

Theorem 7.9. Suppose that $\gamma$ is fine-regular and $\lambda: (\tau \times \sigma)f \rightarrow P(X \times Y)$ is $f_{\gamma}$-compact with respect to $\gamma$ and $\beta$. Let $h: (X, \tau, \tau_f) \rightarrow (Y, \sigma, \sigma_f)$ be a function whose graph $G(h)$ is $f_{\beta}$-closed in $(X \times Y, \tau \times \sigma)$. If a subset $S$ is $f_{\beta}$-compact in $(Y, \sigma, \sigma_f)$, then $h^{-1}(S)$ is $f_{\gamma}$-closed in $(X, \tau, \tau_f)$.

Proof. Suppose that $h^{-1}(S)$ is not $f_{\gamma}$-closed then there exist a point $x$ such that $x \in fc_{\lambda\gamma}(h^{-1}(S))$ and $x \notin h^{-1}(S)$. Since $(x,s) \notin G(h)$ and each $s \in S$ and $fc_{\lambda\gamma}(G(h)) \subseteq G(h)$, there exists a fine-open set $W$ of $(X \times Y, \tau \times \sigma)$ such that $(x,s) \in W$ and $\beta(W) \cap G(h) = \emptyset$. By fine-regularity of $\lambda$, for each $s \in S$ we can take two fine-open sets $U(s)$ and $V(s)$ in $(Y, \sigma, \sigma_f)$ such that $x \in U(s)$, $s \in V(s)$ and $\gamma(U(s)) \times \beta(V(s)) \subseteq \lambda(W)$. Then we have $h(\gamma(U(s))) \cap \beta(V(s)) = \emptyset$. Since $\{V(s) : s \in S\}$ is fine-open cover of $S$, then by $f_{\beta}$-compactness there exists a finite number $s_1, s_2, ..., s_n \in S$ such that $S \subseteq \beta(V(s_1)) \cup \beta(V(s_2)) \cup ... \cup \beta(V(s_n))$. By the fine-regularity of $\gamma$, there exist a fine-open set $U$ such that $x \in U$, $\gamma(U) \subseteq \gamma(U(s_1)) \cap \gamma(U(s_2)) \cap ... \cap \gamma(U(s_n))$. Therefore, we have $\gamma(U) \cap h^{-1}(S) \subseteq U(s_i) \cap h^{-1}(\beta(V(s_i))) = \emptyset$. This shows that $x \notin fc_{\lambda\gamma}(h^{-1}(S))$. This is a contradiction. Therefore, $h^{-1}(S)$ is $f_{\gamma}$-closed.
Theorem 7.10. Suppose that the following condition hold:

(i) $\gamma: \tau_f \to P(X)$ is fine-open

(ii) $\beta: \sigma_f \to P(Y)$ is fine-regular, and

(iii) $\lambda: (\tau \times \sigma)_f \to P(X \times Y)$ is associated with $\gamma$ and $\beta$, and $\lambda$ is fine-regular with respect to $\gamma$ and $\beta$.

Let $h: (X, \tau, \tau_f) \to (Y, \sigma, \sigma_f)$ be a function whose graph $G(h)$ is $f_\lambda$-closed in $(X \times Y, \tau \times \sigma)$. If every cover of $A$ by $f_\gamma$-open sets of $(X, \tau, \tau_f)$ has finite sub cover, then $h(A)$ is $f_\beta$-closed in $(Y, \sigma, \sigma_f)$.

Proof. Similar to Theorem 7.9.

8. Conclusion

In the present paper, the concepts of an operation $\gamma$ on $\tau_f$ are introduced. Also, the concept of $f_\gamma$-open sets are defined, and some of their properties are studied via this operation. Moreover, the concept of $f_\gamma$-closed sets are studied. Furthermore, some types of $f_\gamma$-separation axioms and $f_\gamma \beta$-continuous functions are investigated. In addition, some basic properties of functions with $f_\beta$-closed graphs are obtained.

References


REFERENCES


