On Companion $B$-algebras

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Abstract. This study introduces the concept of companion $B$-algebra and establishes some of its properties. Also, this paper introduces the notions of $\circ$-subalgebra and $\circ$-ideal of a companion $B$-algebra and investigates their relationship. Furthermore, this study establishes some homomorphic properties of $\circ$-subalgebra and $\circ$-ideal.

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1. Introduction


This paper extends the study of $B$-algebras by defining the concept of companion operation and companion $B$-algebras and establishing some of its properties. This study also introduces the concepts of subalgebra and ideal of a companion $B$-algebra and determines some of its homomorphic properties.

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2. Preliminaries

**Definition 2.1.** [11] A \textit{B-algebra} \((X, \ast, 0)\) is a nonempty set \(X\) with a constant 0 and a binary operation “\(\ast\)” satisfying the following axioms: for all \(x, y, z\) in \(X\),

\[
\begin{align*}
(I) & \quad x \ast x = 0, \\
(\text{II}) & \quad x \ast 0 = x, \\
(\text{III}) & \quad (x \ast y) \ast z = x \ast (z \ast (0 \ast y)).
\end{align*}
\]

**Example 2.2.** The set of integers together with the usual subtraction and the constant 0 is a \(B\)-algebra.

**Theorem 2.3.** [11] If \((X, \ast, 0)\) is a \(B\)-algebra, then the following hold: for any \(x, y, z \in X\),

\[
\begin{align*}
(a) & \quad (x \ast y) \ast (0 \ast y) = x \\
(b) & \quad y \ast z = y \ast (0 \ast (0 \ast z)) \\
(c) & \quad x \ast (y \ast z) = (x \ast (0 \ast z)) \ast y \\
(d) & \quad x \ast y = 0 \text{ implies } x = y \\
(e) & \quad 0 \ast x = 0 \ast y \text{ implies } x = y \\
(f) & \quad 0 \ast (0 \ast x) = x.
\end{align*}
\]

**Theorem 2.4.** [13] If \((X, \ast, 0)\) is a \(B\)-algebra, then the following hold: for any \(x, y, z \in X\),

\[
0 \ast (x \ast y) = y \ast x.
\]

**Definition 2.5.** [11] A \(B\)-algebra \((X, \ast, 0)\) is \textit{commutative} if for any \(x, y \in X\),

\[
x \ast (0 \ast y) = y \ast (0 \ast x).
\]

**Theorem 2.6.** [2] Let \((X, \ast, 0)\) be a \(B\)-algebra. If \(x \circ y = x \ast (0 \ast y)\) for all \(x, y \in X\), then \((X, \circ)\) is a group.

**Theorem 2.7.** [11] Let \((G, \circ)\) be a group with identity \(e\). If we define \(x \ast y = x \circ y^{-1}\), then \((G, \ast, e)\) is a \(B\)-algebra.

**Definition 2.8.** [12] Let \((X, \ast, 0)\) be a \(B\)-algebra. A nonempty subset \(H\) of \(X\) is called a \(B\)-subalgebra of \(X\) if \(x \ast y \in H\) for any \(x, y \in H\).

**Definition 2.9.** [5] Let \((X, \ast, 0)\) be a \(B\)-algebra. A nonempty subset \(I\) of \(X\) is called a \(B\)-\textit{ideal} of \(X\) if \(0 \in I\) and \(x \ast y \in I\) and \(y \in I\) imply \(x \in I\).

**Theorem 2.10.** [1] Every subalgebra of a \(B\)-algebra \(X\) is an ideal.

**Definition 2.11.** [10] Let \((A, \ast_A, 0_A)\) and \((B, \ast_B, 0_B)\) be \(B\)-algebras. The mapping \(\phi : A \rightarrow B\) is called a \(B\)-\textit{homomorphism} if \(\phi(x \ast_A y) = \phi(x) \ast_B \phi(y)\) for any \(x, y \in A\). The \textit{kernel} of \(\phi\) is defined as \(\text{Ker}\phi = \{x \in A : \phi(x) = 0_B\}\).
3. Basic Properties of Companion $B$-algebra

**Definition 3.1.** Let $(X, \ast, 0)$ be a $B$-algebra. A binary operation $\odot$ on $X$ is called a subcompanion operation of $X$ if it satisfies for any $x, y \in X$,

$$((x \odot y) \ast x) \ast y = 0$$

(SC)

A subcompanion operation $\odot$ is a companion operation of $X$ if for any $x, y, z \in X$, $(z \ast x) \ast y = 0$ implies $z \ast (x \odot y) = 0$.

**Example 3.2.** Consider the $B$-algebra $(X, \ast, 0)$ with $\ast$ defined below [11]. Define an operation $\odot$ on $X$ as follows:

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<tr>
<th>$\ast$</th>
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By routine calculations, $(X, \ast, \odot, 0)$ is a companion $B$-algebra.

**Example 3.3.** Consider the $B$-algebra $X = (\mathbb{Z}, -, 0)$. Then for all $x, y, z \in \mathbb{Z}$, $((x + y) - x) - y = 0$ and if $(z - x) - y = 0$, then $z - (x + y) = (z - x) - y = 0$. Hence, the binary operation “+” is a companion operation of $\mathbb{Z}$. Therefore, $(\mathbb{Z}, -, +, 0)$ is a companion $B$-algebra.

**Theorem 3.4.** Let $(X, \ast, 0)$ be a $B$-algebra. If $X$ has a companion operation $\odot$, then it is unique.

**Proof:** Assume that the binary operations $\odot_1$ and $\odot_2$ are companion operations on $X$. Then by (SC) applied on $\odot_1$, for any $x, y \in X$, $((x \odot_1 y) \ast x) \ast y = 0$. By (C) applied on $\odot_2$, $(x \odot_1 y) \ast (x \odot_2 y) = 0$. Then by Theorem 2.3(d), $x \odot_1 y = x \odot_2 y$. Thus, $\odot_1 = \odot_2$ and the companion operation is unique.

**Theorem 3.5.** Let $(X, \ast, \odot, 0)$ be a companion $B$-algebra. Let $\ast$ be a binary operation on $X$ such that for all $x, y, z \in X$, $(x \ast y) \ast z = x \ast (y \ast z)$. Then $(X, \ast, \ast, 0)$ is a companion $B$-algebra and $\ast$ is exactly the operation $\odot$.

**Proof:** Suppose $x, y, z \in X$. By hypothesis and Definition 2.1(I), $((x \ast y) \ast x) \ast y = (x \ast y) \ast (x \ast y) = 0$. Hence, $\ast$ is a subcompanion operation. Now, let $(z \ast x) \ast y = 0$. Then by hypothesis, $z \ast (x \ast y) = (z \ast x) \ast y = 0$. Thus, $\ast$ is a companion operation, which is unique by Theorem 3.4. Therefore, $(X, \ast, \ast, 0)$ is a companion $B$-algebra.

**Example 3.6.** Let $X = \{0, 1, 2, 3\}$ be a set with the following table of operations:
Then \((X, *, 0)\) is a \(B\)-algebra [2] and by routine calculations, \((X, *, \odot, 0)\) is a companion \(B\)-algebra. If \(x = 1\) and \(y = 3\), then \(((1 \ast 3) \ast 1) \ast 3 = 2 \neq 0\). Hence, \(\ast\) is not a subcompanion operation and so not a companion operation.

Remark 3.7. If \((X, *, 0)\) is a \(B\)-algebra, then \((X, *, \ast, 0)\) is not always a companion \(B\)-algebra.

In Example 3.6, the condition \(x \ast y = y \ast (0 \ast x)\) does not hold.

Example 3.8. Consider the Klein \(B\)-algebra \(K_4 = \{0, 1, 2, 3\}\) with the following table of operation [4]:

\[
\begin{array}{cccc}
* & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
2 & 2 & 1 & 0 & 3 \\
3 & 3 & 2 & 1 & 0
\end{array}
\]

Then \(x \ast y = y \ast (0 \ast x)\) for any \(x, y \in K_4\) and \((K_4, *, 0)\) is a companion \(B\)-algebra.

The observation in Example 3.8 is generalized in the next theorem.

**Theorem 3.9.** Let \((X, *, 0)\) be a \(B\)-algebra. \(X\) satisfies \(x \ast y = y \ast (0 \ast x)\) for any \(x, y \in X\) if and only if \((X, *, 0)\) is a companion \(B\)-algebra.

**Proof:** Suppose \(x \ast y = y \ast (0 \ast x)\). By Definition 2.1(III), assumption and Definition 2.1(I),

\[
((x \ast y) \ast x) \ast y = (x \ast y) \ast (y \ast (0 \ast x)) = (x \ast y) \ast (x \ast y) = 0.
\]

Suppose \((z \ast x) \ast y = 0\). By Definition 2.1(III), \(z \ast (y \ast (0 \ast x)) = 0\) and by assumption, \(z \ast (x \ast y) = 0\). Therefore, \((X, *, 0)\) is a companion \(B\)-algebra.

Conversely, suppose \((X, *, 0)\) is a companion \(B\)-algebra. By Definition 3.1, \((X, *, 0)\) is a \(B\)-algebra. Let \(x, y \in X\). Then by (SC), \(((x \ast y) \ast x) \ast y = 0\). By Definition 2.1(III),

\[
(x \ast y) \ast (y \ast (0 \ast x)) = 0.
\]

So, \(x \ast y = y \ast (0 \ast x)\) by Theorem 2.3(d). \(\blacksquare\)

**Lemma 3.10.** Let \((X, *, \odot, 0)\) be a companion \(B\)-algebra. Then for any \(x, y, z \in X\), the following hold:

\begin{itemize}
\item[(a)] \(0 \odot y = y\) and \(y \odot 0 = y\);
\item[(b)] \(x \odot y = y \ast (0 \ast x)\);
\item[(c)] if \(x \ast z = y\), then \(x = z \odot y\);
\item[(d)] \(\odot\) is associative in \(X\);
\item[(e)] \(x = (x \odot y) \odot (0 \ast y)\);
\item[(f)] if \((X, *, 0)\) is commutative, then \(x \odot y = x \ast (0 \ast y)\).
\end{itemize}
Proof: Let \((X, *, \circ, 0)\) be a companion \(B\)-algebra and \(x, y, z \in X\).

(a) In (SC), take \(x = 0\), that is, \(0 = ((0 \circ y) * 0) * y = (0 \circ y) * y\). By Theorem 2.3(d), \(0 \circ y = y\). Now, take \(x = y\) and \(y = 0\) in (SC). Then, \(0 = ((y \circ 0) * y) * 0 = (y \circ 0) * y\).

Hence, by Theorem 2.3(d), \(y \circ 0 = y\).

(b) By (SC), \(((x \circ y) * x) * y = 0\). So, by Definition 2.1(III), 
\[(x \circ y) * (y * (0 * x)) = 0\].

Thus, by Theorem 2.3(d), \(x \circ y = y * (0 * x)\).

(c) If \(x * z = y\), then \((x * z) * y = y * y = 0\). By (C), \(x * (z \circ y) = 0\). Hence, by Theorem 2.3(d), \(x = z \circ y\).

(d) By Lemma 3.10(b), Definition 2.1(III) and Theorem 2.3(c), we have
\[
(x \circ y) \circ z = z * (0 * (x \circ y))
= z * (0 * (y * (0 * x)))
= z * ((0 * x) * y)
= (z * (0 * y)) * (0 * x)
= (y \circ z) * (0 * x)
= x \circ (y \circ z).
\]

Thus, the companion operation \(\circ\) is associative.

(e) Note that by Theorem 2.3(f), Definitions 2.1(I), 2.1(III), Theorems 2.3(b) and 2.4, and Lemma 3.10(b),
\[
x = 0 * (0 * x)
= ((0 * y) * (0 * y)) * (0 * x)
= (0 * y) * ((0 * x) * (0 * (0 * y)))
= (0 * y) * ((0 * x) * y)
= (0 * y) * (0 * (y * (0 * x)))
= (0 * y) * (0 * (x \circ y))
= (x \circ y) \circ (0 * y).
\]

(f) Suppose \((X, *, 0)\) is commutative. By Lemma 3.10(b) and Definition 2.5, \(x \circ y = y * (0 * x) = x * (0 * y)\).

Notice that in Example 3.6, \(X\) is commutative and \(1 \circ 1 = 2 \neq 0\). Hence, we have found \(x = 1 \in X\) such that \(x \circ x \neq 0\). Also, \(1 \neq 3 = 0 * 1\). Thus, we have the following remark.

Remark 3.11. If \((X, *, \circ, 0)\) is a companion \(B\)-algebra, then \((X, \circ, 0)\) is not necessarily a \(B\)-algebra.
Proposition 3.12. Suppose \((X, *, \odot, 0)\) is a companion \(B\)-algebra. If \((X, *, 0)\) is a commutative \(B\)-algebra and \(x = 0 * x\) for any \(x \in X\), then \((X, \odot, 0)\) is a \(B\)-algebra.

Proof: Suppose \((X, *, 0)\) is a commutative \(B\)-algebra and \(x, y \in X\). By Lemma 3.10(b), Definition 2.5 and by assumption, \(x \odot y = y * (0 * x) = x * (0 * y) = x * y\). Hence, \((X, \odot, 0) = (X, *, 0)\) is a \(B\)-algebra. \(\blacksquare\)

Example 3.13. Consider the companion \(B\)-algebra in Example 3.2 and consider the following table of operation:

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<th>(\otimes)</th>
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Applying Theorem 2.6, we conclude that \((X, \odot, 0)\) is the group where \(x \otimes y = x *(0 * y)\). Note that \(\odot \neq \otimes\) since \(1 \odot 5 = 4 \neq 3 = 1 \otimes 5\). Thus, by definition of \(\otimes\), \(x \otimes y \neq x \odot y = x *(0 * y)\). Hence, we cannot apply Theorem 2.6 to immediately conclude that \((X, \odot, 0)\) is a group. However, the following theorem says so.

Theorem 3.14. Let \((X, *, \odot, 0)\) be a companion \(B\)-algebra. Then \((X, \odot, 0)\) is a group.

Proof: Note that \(X \neq \emptyset\) since \(0 \in X\). By Lemma 3.10(d) the companion operation \(\odot\) is associative. Note that by Lemma 3.10(a), \(0\) acts as the \(\odot\)-identity element in \((X, \odot, 0)\). Find \(y\) such that \(x \odot y = 0\) and \(y \odot x = 0\). Suppose \(x \odot y = 0\). Then by Lemma 3.10(b), \(y * (0 * x) = 0\). So, by Theorem 2.3(d), \(y = 0 * x\). Also, suppose \(y \odot x = 0\). By Lemma 3.10(b), \(x * (0 * y) = 0\). Then by Theorem 2.3(f), \((0 * (0 * x)) * (0 * y) = 0\) and by Theorem 2.3(d), \(0 * (0 * x) = 0 * y\). Hence, by Theorem 2.3(e), \(y = 0 * x\). Thus, we have found \(x^{-1} = y = 0 * x\) in \((X, \odot, 0)\). Therefore, \((X, \odot, 0)\) is a group. \(\blacksquare\)

Remark 3.15. For any \(x \in X\), \(x^{-1} = 0 * x\) is called the inverse of \(x\) in the group \((X, \odot, 0)\).

Theorem 3.16. Let \((G, \odot)\) be a group with identity \(e\). Then \(G\) determines a companion \(B\)-algebra \((G, *, \odot, e)\) where \(x * y = x \odot y^{-1}\) and \(x \odot y = y \odot x^{-1}\).

Proof: Let \((G, \odot)\) be a group with identity \(e\) and \(x, y \in G\). Define two binary operations \(*\) and \(\odot\) by \(x * y = x \odot y^{-1}\) and \(x \odot y = y \odot x^{-1}\). By Theorem 2.7, \((G, *, e)\) is a \(B\)-algebra. Observe that

\[
((x \odot y) * x) * y = ((x \odot y) \odot x^{-1}) * y \\
= ((x \odot y) \odot x^{-1}) \odot y^{-1} \\
= (x \odot y) \odot (x^{-1} \odot y^{-1}) \\
= (x \odot y) \odot (y \odot x)^{-1}
\]
and 2.1(I), we have

\[
(y \circ x^{-1}) \circ (y \circ x)^{-1} = (y \circ (x^{-1})^{-1}) \circ (y \circ x)^{-1} = (y \circ x) \circ (y \circ x)^{-1} = e.
\]

Hence, \( \otimes \) is a subcompanion operation on \( G \). Suppose \((z \ast x) \ast y = e\). Then \(z \circ (y \circ x)^{-1} = z \circ (x^{-1} \circ y^{-1}) = (z \circ x^{-1}) \circ y^{-1} = (z \ast x) \circ y^{-1} = (z \ast x) \ast y = e\).

Observe that \(z \ast (x \otimes y) = z \ast (y \ast x^{-1}) = z \ast (y \circ (x^{-1})^{-1}) = z \ast (y \circ x) = z \circ (y \circ x)^{-1} = e\).

Hence, \( \otimes \) is a companion operation on \( G \). Thus, \((G, \ast, \otimes, e)\) is a companion \( B \)-algebra. \(\blacksquare\)

Consider the \( B \)-algebra given in Example 3.2. Note that \(X\) is not commutative since there exist \(x = 3\) and \(y = 4\) such that \(3 \ast (0 \ast 4) = 2 \neq 1 = 4 \ast (0 \ast 3)\). Define \(x \circ y = x \ast (0 \ast y)\). If \(x = 3\) and \(y = 2\), then \(((x \circ y) \ast x) \ast y = 2 \neq 0\). Hence, \(\circ\) is not a subcompanion operation.

**Remark 3.17.** If \((X, \ast, 0)\) is a \( B \)-algebra, then \((X, \ast, \circ, 0)\) is not necessarily a companion \( B \)-algebra where the operation \(\circ\) is defined by \(x \circ y = x \ast (0 \ast y)\).

**Example 3.18.** Let \(X = \{0, 1, 2\}\) be a set with the following table of operations, where \(x \circ y = x \ast (0 \ast y)\):

\[
\begin{array}{c|ccc}
* & 0 & 1 & 2 \\
\hline
0 & 0 & 2 & 1 \\
1 & 1 & 0 & 2 \\
2 & 2 & 1 & 0 \\
\end{array}
\quad
\begin{array}{c|ccc}
\circ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\]

By routine calculations, \((X, \ast, 0)\) is a commutative \( B \)-algebra and \((X, \ast, \circ, 0)\) is a companion \( B \)-algebra.

**Theorem 3.19.** If \((X, \ast, 0)\) is a commutative \( B \)-algebra, then \((X, \ast, \circ, 0)\) is a companion \( B \)-algebra where \(x \circ y = x \ast (0 \ast y)\).

**Proof:** Let \((X, \ast, 0)\) be a commutative \( B \)-algebra and \(x, y, z \in X\). Define the operation \(\circ\) by \(x \circ y = x \ast (0 \ast y)\). Note that by Definition 2.1(III), the definition of \(\circ\), Definitions 2.5 and 2.1(I), we have

\[
((x \circ y) \ast x) \ast y = (x \circ y) \ast (y \ast (0 \ast x)) = (x \ast (0 \ast y)) \ast (y \ast (0 \ast x)) = (y \ast (0 \ast x)) \ast (y \ast (0 \ast x)) = 0.
\]

Now, suppose \((z \ast x) \ast y = 0\). Then by the definition of \(\circ\), Definition 2.5 and Definition 2.1(III), \(z \ast (x \circ y) = z \ast (x \ast (0 \ast y)) = z \ast (y \ast (0 \ast x)) = (z \ast x) \ast y = 0\). Hence, \(\circ\) is a companion operation. Therefore, \((X, \ast, \circ, 0)\) is a companion \( B \)-algebra. \(\blacksquare\)
4. On $\circ$-subalgebras

**Definition 4.1.** Let $(X, *, \circ, 0)$ be a companion $B$-algebra and $I$ be a nonempty subset of $X$. Then $I$ is called a $\circ$-subalgebra if $x \circ y \in I$ for any $x, y \in I$.

**Example 4.2.** In Example 3.2, the set $I_1 = \{0, 1, 2\}$ is a $\circ$-subalgebra of $X$, while $I_2 = \{3, 4, 5\}$ is not a $\circ$-subalgebra since $3 \circ 4 = 1 \notin I_2$.

**Theorem 4.3.** Let $(X, *, \circ, 0)$ be a companion $B$-algebra. If $I$ is a $B$-ideal of $X$, then $I$ is a $\circ$-subalgebra of $X$.

**Proof:** Let $(X, *, \circ, 0)$ be a companion $B$-algebra and $I$ be a $B$-ideal of $X$. Then $I \neq \emptyset$. Let $x, y \in I$. By (SC), $((x \circ y) \ast x) \ast y = 0 \in I$. Since $I$ is a $B$-ideal of $X$ and $y \in I$, $(x \circ y) \ast x \in I$ by Definition 2.9. Furthermore, since $x \in I$, $x \circ y \in I$. Therefore, $I$ is a $\circ$-subalgebra of $X$. ■

The converse of Theorem 4.3 need not be true in general. In the companion $B$-algebra $(\mathbb{Z}, -, +, 0)$ in Example 3.3, $I = \mathbb{Z}^+$ is a $\circ$-subalgebra since for all $x, y \in I$, $x + y \in I$. However, $0 \notin I$, thus, $I$ is not a $B$-ideal. Hence, we have the following remark.

**Remark 4.4.** If $I$ is a $\circ$-subalgebra of a companion $B$-algebra $(X, *, \circ, 0)$, then $I$ is not necessarily a $B$-ideal.

Let $(\mathbb{Z}, -, +, 0)$ be the companion $B$-algebra given in Example 3.3. Then $I = \mathbb{Z}^+$ is a $+$-subalgebra. Note that $I_1 = \mathbb{Z}^+ \cup \{0\}$ is a $B$-ideal since $0 \in I_1$. Now, let $x - y \in I_1$ and $y \in I_1$. Then $x - y \geq 0$ and $y \geq 0$. So $x \geq 0$ and $x \in I_1$.

**Theorem 4.5.** Let $(X, *, \circ, 0)$ be a companion $B$-algebra. Suppose $I$ is a $\circ$-subalgebra and $0 \in I$. Then $I$ is a $B$-ideal.

**Proof:** Suppose $I$ is a $\circ$-subalgebra of $X$ and $0 \in I$. Let $u \ast v \in I$ and $v \in I$. Then by Theorem 2.3(a) and Lemma 3.10(b), $u = (u \ast v) \ast (0 \ast v) = v \circ (u \ast v) \in I$. Therefore, $I$ is a $B$-ideal. ■

The following result follows from Theorem 4.3 and Theorem 2.10.

**Corollary 4.6.** Let $(X, *, \circ, 0)$ be a companion $B$-algebra. If $S$ is a $B$-subalgebra of $X$, then $S$ is a $\circ$-subalgebra of $X$.

Consider again the companion $B$-algebra $(\mathbb{Z}, -, +, 0)$ and $+$-subalgebra $I = \mathbb{Z}^+$. Notice that $3 - 5 = -2 \notin I$. Hence, $I$ is not a $B$-subalgebra. Thus, we have the following remark.

**Remark 4.7.** A $\circ$-subalgebra of $X$ is not necessarily a $B$-subalgebra.

**Example 4.8.** Consider Example 3.2 and $\circ$-subalgebra $I = \{0, 1, 2\}$. It is easy to see that $I$ is a $B$-subalgebra and $0 \ast a \in I$, for any $a \in I$.

**Theorem 4.9.** Let $(X, *, \circ, 0)$ be a companion $B$-algebra. Suppose $I$ is a $\circ$-subalgebra and $0 \ast a \in I$, for any $a \in I$. Then $I$ is a $B$-subalgebra.
Proof: Suppose $I$ is a $\odot$-subalgebra and $0 \ast a \in I$, for any $a \in I$. Let $x, y \in I$. Then $0 \ast y \in I$. By Theorem 2.3(b) and Lemma 3.10(b), $x \ast y = x \ast (0 \ast (0 \ast y)) = (0 \ast y) \odot x \in I$. Thus, $I$ is a $B$-subalgebra. 

Consider again the companion $B$-algebra $(\mathbb{Z}, -, +, 0)$ and $+\text{-subalgebra } I = \mathbb{Z}^+$. Take $a = 2$ and $b = 3 \in I$. Then $b^{-1} = 0 - b = -3$ and $a + b^{-1} = -1 \notin I$. Hence, $I$ is not a subgroup of the group $(\mathbb{Z}, +, 0)$. So, we have the following remark.

Remark 4.10. If $I$ is a $\odot$-subalgebra, then $I$ is not necessarily a subgroup.

Consider the companion $B$-algebra $(\mathbb{Z}, -, +, 0)$ and $+\text{-subalgebra } I = \mathbb{Z}^+$. Take $a = 2$ and $b = 3$ \in I. Then $b^{-1} = 0 - b = -3$ and $a + b^{-1} = -1 \notin I$. Hence, $I$ is not a subgroup of the group $(\mathbb{Z}, +, 0)$. So, we have the following remark.

Remark 4.11. The intersection of $\odot$-subalgebras need not be a $\odot$-subalgebra.

The proof of the following theorem is straightforward.

Theorem 4.12. Let $\{I_k : k \in K\}$ be a nonempty collection of $\odot$-subalgebras of a companion $B$-algebra. If $I = \bigcap_{k \in K} I_k \neq \emptyset$, then $I$ is a $\odot$-subalgebra.

Consider Example 3.2. Take $A = \{0,3\}$ and $B = \{0,4\}$. Then $A$ and $B$ are $\odot$-subalgebras. However, $A \cup B = \{0,3,4\}$ is not a $\odot$-subalgebra since $3 \odot 4 = 1 \notin A \cup B$. Hence, we have the following remark.

Remark 4.13. The union of $\odot$-subalgebras need not be a $\odot$-subalgebra.

5. On $\odot$-ideals

Definition 5.1. Let $(X, \ast, \odot, 0)$ be a companion $B$-algebra. A nonempty subset $I$ of $X$ is called a $\odot$-ideal if it satisfies: for any $x, y \in X$,

(i) $0 \in I$ and (ii) $x \odot y \in I$ and $y \in I$ imply $x \in I$.

Example 5.2. In Example 3.2, $\{0,3\}$ is a $\odot$-ideal of $X$. But, $I = \{0,1\}$ is not a $\odot$-ideal since $2 \odot 1 = 0 \in I$ and $1 \in I$ but $2 \notin I$.

Lemma 5.3. Let $(X, \ast, \odot, 0)$ be a companion $B$-algebra and let $I$ be a $\odot$-ideal. If $x \in I$, then $x^{-1} = 0 \ast x \in I$.

Proof: By Remark 3.15, $x^{-1} = 0 \ast x$ is the inverse of $x$. Thus, $(0 \ast x) \odot x = 0 \in I$. Since $x \in I$ and $I$ is a $\odot$-ideal, then $0 \ast x \in I$. 

Theorem 5.4. Let $(X, \ast, \odot, 0)$ be a companion $B$-algebra. If $I$ is a $\odot$-ideal of $X$, then $I$ is a $\odot$-subalgebra.
Proof: Let \( x, y \in I \). Note that by Lemma 5.3, \( 0 \ast y \in I \). Observe that by Lemma 3.10(b), Theorems 2.4, 2.3(c), Definition 2.1(I) and Theorem 2.3(f),

\[
(x \circ y) \circ (0 \ast y) = (y \ast (0 \ast x)) \circ (0 \ast y) \\
= (0 \ast y) \ast (0 \ast (y \ast (0 \ast x))) \\
= (0 \ast y) \ast ((0 \ast x) \ast y) \\
= ((0 \ast y) \ast (0 \ast y)) \ast (0 \ast x) \\
= 0 \ast (0 \ast x) = x.
\]

Since \( x \in I \), \( 0 \ast y \in I \) and \( I \) is a \( \circ \)-ideal, \( x \circ y \in I \). Therefore, \( I \) is a \( \circ \)-subalgebra. ■

The converse of Theorem 5.4 need not be true in general. Note that \( I = \mathbb{Z}^+ \) is a \( \circ \)-subalgebra of \( (\mathbb{Z}, -, +, 0) \) since for all \( x, y \in I \), \( x + y \in I \). However, \( 0 \notin I \). Hence, \( I \) is not a \( \circ \)-ideal. Thus, we have the following remark.

Remark 5.5. If \( I \) is a \( \circ \)-subalgebra, then \( I \) is not necessarily a \( \circ \)-ideal.

Example 5.6. Consider Example 3.6 and \( \circ \)-subalgebra \( I = \{0, 2\} \). Observe that \( 0 \ast 0 = 0 \notin I \) and \( 0 \ast 2 = 2 \in I \), so, \( 0 \ast a \in I \), for any \( a \in I \). It is clear that \( I \) is also a \( \circ \)-ideal.

Theorem 5.7. Let \( (X, \ast, \circ, 0) \) be a companion \( B \)-algebra. Suppose \( I \) is a \( \circ \)-subalgebra of \( X \) and \( 0 \ast a \in I \) for any \( a \in I \). Then \( I \) is a \( \circ \)-ideal.

Proof: Suppose \( I \) is a \( \circ \)-subalgebra and \( 0 \ast a \in I \) for any \( a \in I \). Let \( x \in I \). Then \( 0 \ast x \in I \). Since \( I \) is \( \circ \)-subalgebra, \( 0 = x \circ (0 \ast x) \in I \). Now, suppose \( u \circ v \in I \) and \( v \in I \). Then \( 0 \ast v \in I \). By Lemma 3.10(e), \( u = (u \circ v) \circ (0 \ast v) \). Since \( I \) is a \( \circ \)-subalgebra, \( u \in I \). Therefore, \( I \) is a \( \circ \)-ideal. ■

Theorem 5.8. Let \( (G, \ast, \circ, 0) \) be a companion \( B \)-algebra. A nonempty subset \( I \) of \( G \) is a \( \circ \)-ideal of \( G \) if and only if \( I \) is a subgroup of the group \( (G, \circ, 0) \).

Proof: Let \( I \) be a \( \circ \)-ideal and \( a, b \in I \). By Lemma 5.3, \( b^{-1} = 0 \ast b \in I \). Because \( I \) is also a \( \circ \)-subalgebra by Theorem 5.4, \( a \circ b^{-1} \in I \). Hence, \( I \) is a subgroup.

Conversely, suppose \( I \) is a subgroup of the group \( (G, \circ, 0) \) and \( a, b \in I \). Then \( a \circ b^{-1} \in I \). Note that \( a \circ a^{-1} = 0 \). So, \( 0 \in I \). Suppose \( x \circ y \in I \) and \( y \in I \). Then by Lemma 3.10(e), \( x = (x \circ y) \circ (0 \ast y) = (x \circ y) \circ y^{-1} \in I \). Thus, \( I \) is a \( \circ \)-ideal. ■

The following corollary follows from Theorem 5.8 and 5.4.

Corollary 5.9. Let \( (G, \ast, \circ, 0) \) be a companion \( B \)-algebra. If \( I \) is a subgroup of the group \( (G, \circ, 0) \), then \( I \) is a \( \circ \)-subalgebra.

The following corollary follows from Theorem 5.8.

Corollary 5.10. Let \( \{I_k : k \in K\} \) be a nonempty collection of \( \circ \)-ideals of a companion \( B \)-algebra. If \( I = \bigcap_{k \in K} I_k \neq \emptyset \), then \( I \) is a \( \circ \)-ideal.
Observe that in Example 3.2, \( I_1 = \{0, 3\} \) and \( I_2 = \{0, 4\} \) are \( \circ \)-ideals. But their union, \( I = I_1 \cup I_2 = \{0, 3, 4\} \) is not a \( \circ \)-ideal because \( 1 \circ 4 = 3 \in I \) and \( 4 \in I \) but \( 1 \notin I \). Thus, we have the following remark.

**Remark 5.11.** The union of \( \circ \)-ideals need not be a \( \circ \)-ideal.

### 6. On Companion-\(B\)-homomorphisms

**Definition 6.1.** Let \((X, *_X, \circ_X, 0_X)\) and \((Y, *_Y, \circ_Y, 0_Y)\) be companion \(B\)-algebras. A map \( f : X \to Y \) is called a **companion-\(B\)-homomorphism** if for any \( a, b \in X \),

\[
    f(a *_X b) = f(a) *_Y f(b) \quad \text{and} \quad f(a \circ_X b) = f(a) \circ_Y f(b).
\]

**Example 6.2.** Let \( m \in \mathbb{Z} \) be fixed. The function \( f : \mathbb{Z} \to \mathbb{Z} \) defined by \( f(x) = mx \), \( x \in \mathbb{Z} \), is a companion-\(B\)-homomorphism.

**Remark 6.3.** A companion \(B\)-homomorphism is a \(B\)-homomorphism and a group homomorphism.

**Example 6.4.** Consider the companion \(B\)-algebra \((X, *_1, \circ_1, 0)\) in Example 3.6 and \((Y, *_2, \circ_2, 0)\) in Example 3.8 where \( \circ_2 = *_2 \). Let \( f : X \to Y \) and \( f(x) = \begin{cases} 0, & \text{if } x = 0, 2, \\ 3, & \text{if } x = 1, 3. \end{cases} \)

Then \( f \) is a companion-\(B\)-homomorphism.

**Theorem 6.5.** Suppose \( f : X \to Y \) is a companion \(B\)-homomorphism. Then \( \text{Ker} f \) is a \( \circ \)-subalgebra of \( X \).

**Proof:** Note that by Remark 6.3, \( \text{Ker} f \) is a subgroup of \( X \). Thus, by Corollary 5.9, \( \text{Ker} f \) is also a \( \circ \)-subalgebra.

The proof of the following theorem is straightforward.

**Theorem 6.6.** Suppose \( f : X \to Y \) is a companion \(B\)-homomorphism. If \( I \) is a \( \circ \)-subalgebra of \( X \), then \( f(I) \) is a \( \circ \)-subalgebra of \( Y \).

**Theorem 6.7.** Suppose \( f : X \to Y \) is a companion \(B\)-epimorphism and \( B \) is a \( \circ \)-subalgebra of \( Y \). Then \( f^{-1}(B) \) is a \( \circ \)-subalgebra of \( X \).

**Proof:** Let \( B \subseteq Y \) be a \( \circ \)-subalgebra of \( Y \). Since \( B \neq \emptyset \) and \( f \) is onto, there exist \( a \in B \) and \( x \in X \) such that \( f(x) = a \). Hence, \( x \in f^{-1}(B) \). So, \( f^{-1}(B) \neq \emptyset \). Note that \( f^{-1}(B) = \{ a \in X : f(a) \in B \} \subseteq X \). Now, let \( x, y \in f^{-1}(B) \). Then \( f(x), f(y) \in B \). Because \( B \) is a \( \circ \)-subalgebra, \( f(x \circ y) = f(x) \circ f(y) \in B \). Hence, \( x \circ y \in f^{-1}(B) \). Therefore, \( f^{-1}(B) \) is a \( \circ \)-subalgebra of \( X \).

By Theorem 5.8, a \( \circ \)-ideal is equivalent to a subgroup of \((X, \circ)\). Thus, the following corollary holds:

**Corollary 6.8.** Suppose \( f : X \to Y \) is a companion \(B\)-homomorphism.
(i) If $I$ is a $\odot$-ideal of $X$, then $f(I)$ is a $\odot$-ideal of $Y$.

(ii) If $B \subseteq Y$ is a $\odot$-ideal of $Y$, then $f^{-1}(B)$ is a $\odot$-ideal of $X$.

(iii) $\text{Ker} f$ is a $\odot$-ideal of $X$.

References


