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Bounds for the Topological indices of $\wp$ graph

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Abstract. Topological indices are mathematical measure which correlates to the chemical structures of any simple finite graph. These are used for Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR). In this paper, we define operator graph namely, $\wp$ graph and structured properties. Also, establish the lower and upper bounds for few topological indices namely, Inverse sum indeg index, Geometric-Arithmetic index, Atom-bond connectivity index, first zagreb index and first reformulated Zagreb index of $\wp$-graph.

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Dedicated to Professor H. M. Srivastava on the occasion of his 80th Birth Anniversary

1. Introduction

The topological index thought came out in 1947 from work done by Harold Wiener while he was working on boiling point of paraffin and he named this index as path number, now it is renamed as Wiener index [3]. Since then many other topological indices have been defined and studied. Topological indices are numerical invariants that are associated with the topological characterization of a compound. Topological indices are closely related to the toxicological, physicochemical, pharmacological properties of a chemical compound. The significance of topological indices is mainly related to their utilizing in Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR).

Now we recall some well known topological indices.

In 2010, Vukicevic and Gasperov [8, 10] introduced bond-additive topological index namely, inverse sum indeg index as a significant predictor of total surface area of octane isomers and is defined as

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$$ISI[G] = \sum_{uv\in E[G]} \left[\frac{d_u d_v}{d_u+d_v}\right]$$

D. Vukicevic and B. Furtula (2009) [9] introduced the Geometric-Arithmetic index and is defined as

$$GA[G] = \sum_{uv\in E[G]} \left[\frac{2\sqrt{d_u d_v}}{d_u+d_v}\right]$$

E. Estrada and *et al*, (1990) [1] introduced the atom bond connectivity index and is defined as

$$ABC[G] = \sum_{uv\in E[G]} \left[\sqrt{\frac{d_u+d_v-2}{d_u d_v}}\right]$$

It display an excellent correlation with the heat formation of alkane.

Gutman. I and Trinajstic. N (1972) [2] introduced the first zagreb index and is defined as

$$M_1[G] = \sum_{uv\in E[G]} [d_u + d_v]$$

A. Milicevic *et al.* (2004) [7] introduced the first reformulated zagreb index and is defined as

$$EM_1[G] = \sum_{uv\in E[G]} [d_u + d_v - 2]^2$$

Definition 1.1. *[4] The jump-graph $J(G)$ of a graph G is the graph defined on $E(G)$ where two vertices are adjacent if and only if their corresponding edges are not adjacent in G.*

Definition 1.2. *[6] The corona product of $G \odot H$ of these two graphs is obtained by taking one copy of G and n_1 copies of H and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, where $1 \leq i \leq n_1$.*

In order to study bounds on $\wp$-graph, we divide the paper into few sections. The section one contains preliminaries, definitions of well known topological indices which are useful to prove our main results. Section two deals with new class of operator graph with their properties. Section three consisting of results related to bounds for defined class of graph using recalled topological indices. Paper conclude with the conclusion and references.

Let G and H be graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$ respectively. The degree of vertex v is the number of vertices adjacent to v.

Let $\{V(G) \bigcap V(H) = \emptyset | g \in V(G), h \in V(H)\}$. The number of vertices and number of edges in the graphs G and H are represented by n_1 , n_2 and m_1 , m_2 respectively. We have

$$\Delta_G \geq deg_G(g), \qquad \delta_G \leq deg_G(g)$$
$$\Delta_H \geq deg_H(h), \qquad \delta_H \leq deg_H(h)$$

The bounds for different topological indices are obtained by many researchers [5].

Now we will define a new class of operator graph namely J-vertex corona product of graph ($\wp$-graph).

Definition 1.3. *The $J(G) \odot H = \wp$ is a graph obtained from one copy of graph $J(G)$ and m_1 copies of H and joining a vertex of $V[J(G)]$, that is, on the i^{th} position in $J(G)$ to every vertex in the i^{th} copy of H.*

Properties of $\wp$-graph:

The $\wp$-graph has

(i). $m_1 + m_1 n_2$ vertices

(ii). $m_1[m_2 + n_2] + \frac{m_1[m_1-1]}{2} - \sum_{uv \in E(G)} \left[\frac{deg_G u + deg_G v - 2}{2}\right]$ edges.

(iii). The degree of a vertex $v \in V[\wp]$ is given by

$$deg_{\wp} u = \begin{cases} deg_H u + 1, & \text{if } u \in V[H] \\ deg_{J(G)} u + n_2, & \text{if} u \in V[J[G]]. \end{cases}$$

2. Bounds on various topological indices of $\wp$ graph

In this section, we formulate the bounds on the ISI, GA, ABC, M_1 and EM_1 indices of $\wp$-graph.

Theorem 1. *Let G and H are two simple connected graphs, then the bounds for the inverse sum indeg index of $\wp$ are given by*

$$\begin{aligned} ISI[\wp] \geq & \frac{m_1 m_2(\Delta_H + 1)}{2} + m_1 n_2 \left[\frac{(\Delta_H + 1).[(m_1 - 1) - 2[\Delta_G - 1] + n_2]}{\Delta_H + m_1 - 2[\Delta_G - 1] + n_2}\right] \\ & + \left[\frac{m_1(m_1 - 1)}{2} - m_1(\Delta_G - 1)\right]\left[\frac{(m_1 - 1) - 2[\Delta_G - 1] + n_2}{2}\right] \end{aligned}$$

and

$$\begin{aligned} ISI[\wp] \leq & \frac{m_1 m_2(\delta_H + 1)}{2} + m_1 n_2 \left[\frac{(\delta_H + 1).[(m_1 - 1) - 2[\delta_G - 1] + n_2]}{\delta_H + m_1 - 2[\delta_G - 1] + n_2}\right] \\ & + \left[\frac{m_1(m_1 - 1)}{2} - m_1(\delta_G - 1)\right]\left[\frac{(m_1 - 1) - 2[\delta_G - 1] + n_2}{2}\right]. \end{aligned}$$

Proof.

Consider,

$$ISI[\wp] = m_1 \sum_{uv\in E(H)} \left[\frac{(deg_H u+1).(deg_H v+1)}{(deg_H u+1)+(deg_H v+1)}\right] + \sum_{ef\in E(J[G])} \left[\frac{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)}\right]$$

$$+ \sum_{e\in V(J[G])} \sum_{u\in V(H)} \left[\frac{(deg_H u+1).(deg_{J(G)}e+n_2)}{(deg_H u+1)+(deg_{J(G)}e+n_2)}\right]$$

$$= m_1 m_2 \left[\frac{(deg_H u+1).(deg_H v+1)}{(deg_H u+1)+(deg_H v+1)}\right] + m_1 n_2 \left[\frac{(deg_H u+1).(deg_{J[G]}e+n_2)}{(deg_H u+1)+(deg_{J[G]}e+n_2)}\right]$$

$$+ \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{deg_G u+deg_G v-2}{2}\right]\right]\left[\frac{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)}\right]$$

$$= m_1 m_2 \left[\frac{(deg_H u+1).(deg_H v+1)}{(deg_H u+1)+(deg_H v+1)}\right] + \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{deg_G u+deg_G v-2}{2}\right]\right]$$

$$\left[\frac{[(m_1-1)-[deg_G u+deg_G v-2]+n_2].[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}{[(m_1-1)-[deg_G u+deg_G v-2]+n_2]+[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}\right]$$

$$+ m_1 n_2 \left[\frac{(deg_H u+1).[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}{(deg_H u+1)+[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}\right]$$

$$\geq m_1 m_2 \left[\frac{(\Delta_H+1).(\Delta_H+1)}{(\Delta_H+1)+(\Delta_H+1)}\right] + m_1 n_2 \left[\frac{(\Delta_H+1).[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}{(\Delta_H+1)+[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}\right]$$

$$+ \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{\Delta_G+\Delta_G-2}{2}\right]\right]$$

$$\left[\frac{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2].[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]+[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}\right]$$

$$\geq \frac{m_1 m_2(\Delta_H+1)}{2} + m_1 n_2 \left[\frac{(\Delta_H+1).[(m_1-1)-2[\Delta_G-1]+n_2]}{\Delta_H+1+m_1-1-2[\Delta_G-1]+n_2}\right]$$

$$+ \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{2(\Delta_G-1)}{2}\right]\right]\left[\frac{(m_1-1)-2[\Delta_G-1]+n_2}{2}\right].$$

$$ISI[\wp] \geq \frac{m_1 m_2(\Delta_H+1)}{2} + m_1 n_2 \left[\frac{(\Delta_H+1).[(m_1-1)-2[\Delta_G-1]+n_2]}{\Delta_H+m_1-2[\Delta_G-1]+n_2}\right]$$
$$+ \left[\frac{m_1(m_1-1)}{2} - m_1(\Delta_G-1)\right]\left[\frac{(m_1-1)-2[\Delta_G-1]+n_2}{2}\right].$$

Similarly,

$$ISI[\wp] \leq \frac{m_1 m_2(\delta_H+1)}{2} + m_1 n_2 \left[\frac{(\delta_H+1).[(m_1-1)-2[\delta_G-1]+n_2]}{\delta_H+m_1-2[\delta_G-1]+n_2}\right]$$

$$+\left[\frac{m_1(m_1-1)}{2}-m_1(\delta_G-1)\right]\left[\frac{(m_1-1)-2[\delta_G-1]+n_2}{2}\right].$$

Theorem 2. *Let G and H are two simple connected graphs, then the bounds for the geometric-arithmetic index of $\wp$ are given by*

$$GA[\wp]\geq m_1m_2+\frac{2m_1n_2\sqrt{(\Delta_H+1).[(m_1-1)-2[\Delta_G-1]+n_2]}}{\Delta_H+m_1-2[\Delta_G-1]+n_2}+\frac{m_1(m_1-1)}{2}-m_1(\Delta_G-1)$$

and

$$GA[\wp]\leq m_1m_2+\frac{2m_1n_2\sqrt{(\delta_H+1).[(m_1-1)-2[\delta_G-1]+n_2]}}{\delta_H+m_1-2[\delta_G-1]+n_2}+\frac{m_1(m_1-1)}{2}-m_1(\delta_G-1).$$

Proof.

Consider $GA[\wp]=m_1\sum_{uv\in E(H)}\left[\frac{2\sqrt{(deg_Hu+1).(deg_Hv+1)}}{(deg_Hu+1)+(deg_Hv+1)}\right]$

$$+\sum_{ef\in E(J[G])}\left[\frac{2\sqrt{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}}{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)}\right]$$

$$+\sum_{e\in V(J[G])}\sum_{u\in V(H)}\left[\frac{2\sqrt{(deg_Hu+1).(deg_{J(G)}e+n_2)}}{(deg_Hu+1)+(deg_{J(G)}e+n_2)}\right]$$

$$=m_1m_2\left[\frac{2\sqrt{(deg_Hu+1).(deg_Hv+1)}}{(deg_Hu+1)+(deg_Hv+1)}\right]+m_1n_2\left[\frac{2\sqrt{(deg_Hu+1).(deg_{J[G]}e+n_2)}}{(deg_Hu+1)+(deg_{J[G]}e+n_2)}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_Gu+deg_Gv-2}{2}\right]\right]\left[\frac{2\sqrt{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}}{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)}\right]$$

$$=m_1m_2\left[\frac{2\sqrt{(deg_Hu+1).(deg_Hv+1)}}{(deg_Hu+1)+(deg_Hv+1)}\right]$$

$$+m_1n_2\left[\frac{2\sqrt{(deg_Hu+1).[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]}}{(deg_Hu+1)+[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_Gu+deg_Gv-2}{2}\right]\right]$$

$$\left[\frac{2\sqrt{[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2].[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]}}{[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]+[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]}\right]$$

$$\geq m_1m_2\left[\frac{2\sqrt{(\Delta_H+1).(\Delta_H+1)}}{(\Delta_H+1)+(\Delta_H+1)}\right]+\left[\frac{2m_1n_2\sqrt{(\Delta_H+1).[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}}{\Delta_H+1+m_1-1-[\Delta_G+\Delta_G-2]+n_2}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{\Delta_G+\Delta_G-2}{2}\right]\right]$$
$$\left[\frac{2\sqrt{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2].[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}}{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]+[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}\right].$$

$$GA[\wp]\geq m_1m_2+\frac{2m_1n_2\sqrt{(\Delta_H+1).[(m_1-1)-2[\Delta_G-1]+n_2]}}{\Delta_H+m_1-2[\Delta_G-1]+n_2}+\frac{m_1(m_1-1)}{2}-m_1(\Delta_G-1).$$

Similarly,

$$GA[\wp]\leq m_1m_2+\frac{2m_1n_2\sqrt{(\delta_H+1).[(m_1-1)-2[\delta_G-1]+n_2]}}{\delta_H+m_1-2[\delta_G-1]+n_2}+\frac{m_1(m_1-1)}{2}-m_1(\delta_G-1).$$

Theorem 3. *Let G and H are two simple connected graphs, then the bounds for the atom-bond connective index of $\wp$ are given by*

$$ABC[\wp]\geq m_1m_2\left[\sqrt{\frac{2\Delta_H}{(\Delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\Delta_H+m_1-2\Delta_G+n_2}{[\Delta_H+1].[m_1-1-2[\Delta_G-1]+n_2]}}\right]$$
$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\Delta_G-1)\right]\left[\sqrt{\frac{2[m_1-2\Delta_G+n_2]}{[(m_1-1)-2[\Delta_G-1]+n_2]^2}}\right]$$

and

$$ABC[\wp]\leq m_1m_2\left[\sqrt{\frac{2\delta_H}{(\delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\delta_H+m_1-2\delta_G+n_2}{[\delta_H+1].[m_1-1-2[\delta_G-1]+n_2]}}\right]$$
$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\delta_G-1)\right]\left[\sqrt{\frac{2[m_1-2\delta_G+n_2]}{[(m_1-1)-2[\delta_G-1]+n_2]^2}}\right].$$

Proof.

Consider

$$ABC[\wp]=m_1\sum_{uv\in E(H)}\left[\sqrt{\frac{(deg_Hu+1)+(deg_Hv+1)-2}{(deg_Hu+1).(deg_Hv+1)}}\right]$$
$$+\sum_{e\in V(J[G])}\sum_{u\in V(H)}\left[\sqrt{\frac{(deg_Hu+1)+(deg_{J(G)}e+n_2)-2}{(deg_Hu+1).(deg_{J(G)}e+n_2)}}\right]$$
$$+\sum_{ef\in E(J[G])}\left[\sqrt{\frac{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)-2}{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}}\right]$$
$$=m_1m_2\left[\sqrt{\frac{(deg_Hu+1)+(deg_Hv+1)-2}{(deg_Hu+1).(deg_Hv+1)}}\right]+m_1n_2\left[\sqrt{\frac{(deg_Hu+1)+(deg_{J[G]}e+n_2)-2}{(deg_Hu+1).(deg_{J[G]}e+n_2)}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_G u+deg_G v-2}{2}\right]\right]\left[\sqrt{\frac{(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)-2}{(deg_{J[G]}e+n_2).(deg_{J[G]}f+n_2)}}\right]$$

$$=m_1m_2\left[\sqrt{\frac{(deg_H u+1)+(deg_H v+1)-2}{(deg_H u+1).(deg_H v+1)}}\right]+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_G u+deg_G v-2}{2}\right]\right]$$

$$\left[\sqrt{\frac{[(m_1-1)-[deg_G u+deg_G v-2]+n_2]+[(m_1-1)-[deg_G u+deg_G v-2]+n_2]-2}{[(m_1-1)-[deg_G u+deg_G v-2]+n_2].[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}}\right]$$

$$+m_1n_2\left[\sqrt{\frac{(deg_H u+1)+[(m_1-1)-[deg_G u+deg_G v-2]+n_2]-2}{(deg_H u+1).[(m_1-1)-[deg_G u+deg_G v-2]+n_2]}}\right]$$

$$\geq m_1m_2\left[\sqrt{\frac{(\Delta_H+1)+(\Delta_H+1)-2}{(\Delta_H+1).(\Delta_H+1)}}\right]+m_1n_2\left[\sqrt{\frac{\Delta_H+1+m_1-1-[\Delta_G+\Delta_G-2]+n_2-2}{[\Delta_H+1].[m_1-1-[\Delta_G+\Delta_G-2]+n_2]}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{\Delta_G+\Delta_G-2}{2}\right]\right]$$

$$\left[\sqrt{\frac{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]+[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]-2}{[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2].[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]}}\right]$$

$$\geq m_1m_2\left[\sqrt{\frac{2(\Delta_H+1)-2}{(\Delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\Delta_H+m_1-2[\Delta_G-1]+n_2-2}{[\Delta_H+1].[m_1-1-2[\Delta_G-1]+n_2]}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{2(\Delta_G-1)}{2}\right]\right]\left[\sqrt{\frac{2[(m_1-1)-2[\Delta_G-1]+n_2]-2}{[(m_1-1)-2[\Delta_G-1]+n_2]^2}}\right]$$

$$\geq m_1m_2\left[\sqrt{\frac{2(\Delta_H+1-1)}{(\Delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\Delta_H+m_1-2[\Delta_G-1+1]+n_2}{[\Delta_H+1].[m_1-1-2[\Delta_G-1]+n_2]}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\Delta_G-1)\right]\left[\sqrt{\frac{2[m_1-1-2[\Delta_G-1]+n_2-1]}{[(m_1-1)-2[\Delta_G-1]+n_2]^2}}\right].$$

$$ABC[\wp]\geq m_1m_2\left[\sqrt{\frac{2\Delta_H}{(\Delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\Delta_H+m_1-2\Delta_G+n_2}{[\Delta_H+1].[m_1-1-2[\Delta_G-1]+n_2]}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\Delta_G-1)\right]\left[\sqrt{\frac{2[m_1-2\Delta_G+n_2]}{[(m_1-1)-2[\Delta_G-1]+n_2]^2}}\right].$$

Similarly,

$$ABC[\wp]\leq m_1m_2\left[\sqrt{\frac{2\delta_H}{(\delta_H+1)^2}}\right]+m_1n_2\left[\sqrt{\frac{\delta_H+m_1-2\delta_G+n_2}{[\delta_H+1].[m_1-1-2[\delta_G-1]+n_2]}}\right]$$

$$+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\delta_G-1)\right]\left[\sqrt{\frac{2[m_1-2\delta_G+n_2]}{[(m_1-1)-2[\delta_G-1]+n_2]^2}}\right].$$

Theorem 4. *Let G and H are two simple connected graphs, then the bounds for the first zagreb index of $\wp$ are given by*

$$\begin{aligned}M_1[\wp]&\geq 2m_1m_2(\Delta_H+1)+m_1n_2[\Delta_H+m_1-2[\Delta_G-1]+n_2]\\&+2\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\Delta_G-1)\right][m_1-1-2[\Delta_G-1]+n_2]\end{aligned}$$

and

$$\begin{aligned}M_1[\wp]&\leq 2m_1m_2(\delta_H+1)+m_1n_2[\delta_H+m_1-2[\delta_G-1]+n_2]\\&+2\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1(\delta_G-1)\right][m_1-1-2[\delta_G-1]+n_2].\end{aligned}$$

Proof.

$$\begin{aligned}\text{Consider } M_1[\wp]&=m_1\sum_{uv\in E(H)}[(deg_Hu+1)+(deg_Hv+1)]+\sum_{e\in V(J[G])}\sum_{u\in V(H)}[(deg_Hu+1)+(deg_{J(G)}e+n_2)]\\
&+\sum_{ef\in E(J[G])}[(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)]\\
&=m_1m_2[(deg_Hu+1)+(deg_Hv+1)]+m_1n_2[(deg_Hu+1)+(deg_{J[G]}e+n_2)]\\
&+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_Gu+deg_Gv-2}{2}\right]\right][(deg_{J[G]}e+n_2)+(deg_{J[G]}f+n_2)]\\
&=m_1m_2[(deg_Hu+1)+(deg_Hv+1)]+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{deg_Gu+deg_Gv-2}{2}\right]\right]\\
&[[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]+[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]]\\
&+m_1n_2[(deg_Hu+1)+[(m_1-1)-[deg_Gu+deg_Gv-2]+n_2]]\\
&\geq m_1m_2[(\Delta_H+1)+(\Delta_H+1)]+m_1n_2[\Delta_H+1+m_1-1-[\Delta_G+\Delta_G-2]+n_2]\\
&+\left[\left[\frac{m_1(m_1-1)}{2}\right]-m_1\left[\frac{\Delta_G+\Delta_G-2}{2}\right]\right][2[(m_1-1)-[\Delta_G+\Delta_G-2]+n_2]]\\
&\geq 2m_1m_2(\Delta_H+1)+m_1n_2[\Delta_H+m_1-2[\Delta_G-1]+n_2]\\
&+2\left[\frac{m_1(m_1-1)}{2}-m_1\left[\frac{2(\Delta_G-1)}{2}\right]\right][m_1-1-2[\Delta_G-1]+n_2].\end{aligned}$$

$$M_1[\wp] \geq 2m_1m_2(\Delta_H+1) + m_1n_2[\Delta_H + m_1 - 2[\Delta_G - 1] + n_2]$$
$$+ 2\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\Delta_G - 1)\right][m_1 - 1 - 2[\Delta_G - 1] + n_2].$$

Similarly,

$$M_1[\wp] \leq 2m_1m_2(\delta_H+1) + m_1n_2[\delta_H + m_1 - 2[\delta_G - 1] + n_2]$$
$$+ 2\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\delta_G - 1)\right][m_1 - 1 - 2[\delta_G - 1] + n_2].$$

Theorem 5. *Let G and H are two simple connected graphs, then the bounds for the first reformulated zagreb index of $\wp$ are given by*

$$EM_1[\wp] \geq 4m_1m_2\Delta_H^2 + m_1n_2[\Delta_H + m_1 - 2\Delta_G + n_2]^2$$
$$+ 4\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\Delta_G - 1)\right][m_1 - 2\Delta_G + n_2]^2$$

and

$$EM_1[\wp] \leq 4m_1m_2\delta_H^2 + m_1n_2[\delta_H + m_1 - 2\delta_G + n_2]^2$$
$$+ 4\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\delta_G - 1)\right][m_1 - 2\delta_G + n_2]^2.$$

Proof.

Consider, $EM_1[\wp] = m_1 \sum_{uv\in E(H)} [(deg_H u + 1) + (deg_H v + 1) - 2]^2$

$$+ \sum_{e\in V(J[G])} \sum_{u\in V(H)} [(deg_H u + 1) + (deg_{J(G)} e + n_2) - 2]^2$$
$$+ \sum_{ef\in E(J[G])} [(deg_{J[G]} e + n_2) + (deg_{J[G]} f + n_2) - 2]^2$$
$$= m_1m_2[(deg_H u + 1) + (deg_H v + 1) - 2]^2 + m_1n_2[(deg_H u + 1) + (deg_{J[G]} e + n_2) - 2]^2$$
$$+ \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{deg_G u + deg_G v - 2}{2}\right]\right][(deg_{J[G]} e + n_2) + (deg_{J[G]} f + n_2) - 2]^2$$
$$= m_1m_2[(deg_H u + 1) + (deg_H v + 1) - 2]^2 + \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{deg_G u + deg_G v - 2}{2}\right]\right]$$
$$[[(m_1 - 1) - [deg_G u + deg_G v - 2] + n_2] + [(m_1 - 1) - [deg_G u + deg_G v - 2] + n_2] - 2]^2$$
$$+ m_1n_2[(deg_H u + 1) + [(m_1 - 1) - [deg_G u + deg_G v - 2] + n_2] - 2]^2$$
$$\geq m_1m_2[(\Delta_H + 1) + (\Delta_H + 1) - 2]^2 + \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{\Delta_G + \Delta_G - 2}{2}\right]\right]$$
$$[[(m_1 - 1) - [\Delta_G + \Delta_G - 2] + n_2] + [(m_1 - 1) - [\Delta_G + \Delta_G - 2] + n_2] - 2]^2$$

$$+ m_1n_2[\Delta_H + 1 + m_1 - 1 - [\Delta_G + \Delta_G - 2] + n_2 - 2]^2$$
$$\geq m_1m_2[2(\Delta_H + 1) - 2]^2 + \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1\left[\frac{2(\Delta_G-1)}{2}\right]\right]$$
$$[2[(m_1 - 1) - 2[\Delta_G - 1] + n_2] - 2]^2 + m_1n_2[\Delta_H + m_1 - 2[\Delta_G - 1] + n_2 - 2]^2$$

$$\geq 4m_1m_2[\Delta_H + 1 - 1]^2 + \left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\Delta_G - 1)\right][2[m_1 - 1 - 2[\Delta_G - 1] + n_2 - 1]]^2$$
$$+ m_1n_2[\Delta_H + m_1 - 2[\Delta_G - 1 + 1] + n_2]^2$$
$$\geq 4m_1m_2\Delta_H^2 + m_1n_2[\Delta_H + m_1 - 2\Delta_G + n_2]^2 + 4\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\Delta_G - 1)\right]$$
$$[m_1 - 2\Delta_G + n_2]^2.$$
$$EM_1[\wp] \geq 4m_1m_2\Delta_H^2 + m_1n_2[\Delta_H + m_1 - 2\Delta_G + n_2]^2 + 4\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\Delta_G - 1)\right]$$
$$[m_1 - 2\Delta_G + n_2]^2.$$

Similarly,

$$EM_1[\wp] \leq 4m_1m_2\delta_H^2 + m_1n_2[\delta_H + m_1 - 2\delta_G + n_2]^2 + 4\left[\left[\frac{m_1(m_1-1)}{2}\right] - m_1(\delta_G - 1)\right]$$
$$[m_1 - 2\delta_G + n_2]^2.$$

3. Conclusion

In this work, we considered $\wp$-graph and concentrated five important topological indices and determine their bounds. Similar way, researchers can considering different class of topological indices and determine their corresponding bounds for $\wp$-graph.

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