Existence and uniqueness of solutions for the nonlinear fractional differential equations with two-point and integral boundary conditions

Y.A. Sharifov\textsuperscript{1,2,*}, S.A. Zamanova\textsuperscript{3}, R.A. Sardarova\textsuperscript{3}

\textsuperscript{1} Institute of Mathematics and Mechanics, ANAS, Baku, Azerbaijan
\textsuperscript{2} Baku State University Baku, Azerbaijan
\textsuperscript{3} Azerbaijan State University of Economics (UNECE), Baku, Azerbaijan

Abstract. In this paper the existence and uniqueness of solutions to the fractional differential equations with two-point and integral boundary conditions is investigated. The Green function is constructed, and the problem under consideration is reduced to the equivalent integral equation. Existence and uniqueness of a solution to this problem is analyzed using the Banach the contraction mapping principle and Krasnoselski\u0161 fixed point theorem.

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1. Introduction

In recent years, the theory of the fractional differential equations has played a very important role in a new branch of applied mathematics, which has been utilized for mathematical models in engineering, physics, chemistry, signal analysis, etc. For details and applications, we refer the reader to the classical reference texts such as [1-6]. Fractional differential equations are considered as a valuable tool to model many real world problems. Boundary value problems for such differential equations represent an important class of applied analysis. Most of the studied fractional differential equations by taking Caputo or Riemann-Liouville derivatives. Engineers and scientists have developed some new models that involve fractional differential equations for which the Riemann-Liouville derivative is not considered appropriate. Therefore, certain modifications were introduced to avoid the difficulties and some new types of fractional order derivative operators were introduced in the literature by authors like Caputo, Hadamard, and Erdely Kober, etc.

*Corresponding author.
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Email addresses: sharifov22@rambler.ru (Y.A. Sharifov), sevinc.zamanova@gmail.com (S.A. Zamanova), sardarova.rita.77@gmail.com (R.A. Sardarova)
Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems presenting both theoretical and practical importance. They include two, three, multipoint and nonlocal boundary value problems as special cases. Integral boundary value problems occur in the mathematical modeling of variety of physics processes and have recently received considerable attention. For some recent works on the boundary value problems with integral boundary conditions we refer to [7-15, 17-19, 21-25] and the references cited therein.

In this paper, we study existence and uniqueness of nonlinear fractional differential equations of the type

$$cD_0^\alpha x(t) = f(t, x(t)), \text{ for } t \in [0, T],$$

subject to two-point and integral boundary conditions

$$Ax(0) + \int_0^T n(t)x(t)\,dt + Bx(T) = C,$$

where $0 < \alpha < 1, cD_0^\alpha$ is the Caputo fractional derivatives, $A, B \in \mathbb{R}^{n \times n}$ and $n(t) : [0, T] \to \mathbb{R}^{n \times n}$ are given matrices and $\det \left(A + \int_0^T n(t)\,dt + B\right) \neq 0$.

The paper is organized as follows. In Section 2, we give some notations, recall some concepts, and introduce a concept of a continuous solution for our problem. In Section 3, we give two main results: the first result based on the Banach contraction principle and the second result based on the Krasnoselski’s fixed point theorem.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper. By $C([0, T] ; \mathbb{R}^n)$ we denote the Banach space of all continuous functions from $[0, T]$ into $\mathbb{R}^n$ with the norm $\|x\| = \max \{|x(t)| : t \in [0, T]\}$, where $|\cdot|$ norm in $\mathbb{R}^n$.

**Definition 1.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $y : [0, \infty) \to \mathbb{R}$, is defined by

$$(J^\alpha y)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} y(s)\,ds,$$

provided the right-hand side exists on $(0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function defined for any complex number $z$ as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$
Definition 2. The (right-sided) Riemann-Liouville fractional derivative is defined by
\[ RL D^\alpha y = (D^\alpha y)(x) = \frac{d}{dx^n} (J^{n-\alpha}y)(x), \quad x > 0, \]
where \( n = \lceil \alpha \rceil + 1 \), denotes the integer part of the real number \( \alpha \), provided the right-hand side is point-wise defined on \((0, \infty)\).

The Riemann-Liouville fractional derivative is left-inverse (but not right-inverse) of the Riemann-Liouville fractional integral, which is a natural generalization of the Cauchy formula for the \( n \)-fold primitive of a function \( y \). As to the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, they should be given as (bounded) initial values of the fractional integral \( J^{n-\alpha} \) and of its integer derivatives of order \( k = 1, 2, \ldots, n-1 \).

Definition 3. The Caputo fractional derivative of order \( \alpha > 0 \) of a continuous function \( y \), is defined by
\[ CD^\alpha 0^+ y = (D^\alpha y)(x) = \left( J^{n-\alpha}y^{(n)} \right)(x), \quad n - 1 < \alpha \leq n, \quad x > 0, \]
provided the right-hand side is point-wise defined on \((a, \infty)\).

Obviously, this definition allows one to consider the initial-value problems for the fractional differential equations with initial conditions that are expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order.

Remark 1. \([20]\) Under natural conditions on \( y(x) \), the Caputo fractional derivative becomes the conventional integer order derivative of the function \( y(x) \) as \( \alpha \to n \).

Remark 2. \([20]\) Let \( \alpha, \beta > 0 \) and \( n = \lceil \alpha \rceil + 1 \); then the following relations hold:
\[ c D^\alpha 0^+ t^\beta = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} t^{\beta - \alpha}, \beta > n, \]
\[ c D^\alpha 0^+ t^k = 0, \quad k = 0, 1, \ldots, n - 1. \]

Lemma 1. \([20]\) For \( \alpha > 0, y(t) \in C([0,T]) \cap L_1([0,T]) \) the homogeneous fractional differential equation
\[ c D^\alpha 0^+ y(t) = 0, \]
has a solution
\[ y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \]
where \( c_i \in \mathbb{R}, i = 1, 2, \ldots, n - 1 \) and \( n = \lceil \alpha \rceil + 1 \).

Lemma 2. \([20]\) Assume that \( y(t) \in C([0,T]) \cap L_1([0,T]), \) with derivative of order \( n \) that belongs to \( C([0,T]) \cap L_1([0,T]) \). Then
\[ I^\alpha c D^\alpha 0^+ y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \]
where \( c_i \in \mathbb{R}, i = 1, 2, \ldots, n - 1 \) and \( n = \lceil \alpha \rceil + 1 \).
Lemma 3. [20] Let $p, q \geq 0, f \in L^1 ([0, T])$. Then
\[
I_{0+}^p I_{0+}^q f (t) = I_{0+}^{p+q} f (t) = I_{0+}^q I_{0+}^p f (t)
\]
is satisfied almost everywhere on $[0, T]$. Moreover, if $f \in C ([0, T])$, then (14) is true for all $t \in [0, T]$.

Lemma 4. [20] If $\alpha > 0, f \in C ([0, T])$, then
\[
c D_0^\alpha I_{0+}^\alpha f (t) = f (t)
\]
for all $t \in [0, T]$.

We have the following result which is useful in what follows.

Theorem 1. Let $y \in C ([0, T] ; R^n)$. Then the unique solution of the linear boundary value problem
\[
\begin{align*}
\frac{c D_0^\alpha x (t)}{T} &= y(t), \\
Ax(0) + \int_0^t n(t)x(t)dt + Bx(T) &= C,
\end{align*}
\]
is given by
\[
x (t) = N^{-1}C + \frac{1}{\Gamma (\alpha)} \int_0^t (t-s)^{\alpha-1} y (s) 
\]
\[
- \frac{N^{-1}}{\Gamma (\alpha)} \int_0^t n (t) \int_0^t (t-s)^{\alpha-1} y (s) 
\]
\[
dt - \frac{N^{-1}B}{\Gamma (\alpha)} \int_0^T (T-s)^{\alpha-1} y (s) 
\]
dsdt,
\]
where
\[
N = \left( A + \int_0^T n(t)dt + B \right).
\]

Proof. Assume that $x$ is a solution of boundary value problem (3). Then we have
\[
x (t) = x (0) + \frac{1}{\Gamma (\alpha)} \int_0^t (t-s)^{\alpha-1} y (s) 
\]
\[
ds, \quad t \in [0, T],
\]
where $x(0)$ is still an arbitrary constant vector.

For determining $x(0)$ we use the boundary value condition $Ax(0) + \int_0^T n(t)x(t)dt + Bx(T) = C$ :
\[
C = Ax (0) + \int_0^T n(t)x(t)dt + Bx(T) = \left( A + \int_0^T n(t)dt + B \right) x(0)
\]
\[
+ \frac{1}{\Gamma (\alpha)} \int_0^t n (t) \int_0^t (t-s)^{\alpha-1} y (s) 
\]
dsdt + \frac{B}{\Gamma (\alpha)} \int_0^T (T-s)^{\alpha-1} y (s) 
\]
ds.
From here we get
\[
x(0) = N^{-1}C - \frac{N^{-1}}{\Gamma(\alpha)} \int_{0}^{T} n(t) \left( t - s \right)^{\alpha-1} y(s) \, ds \, dt + \frac{N^{-1}B}{\Gamma(\alpha)} \int_{0}^{T} (T - s) y(s) \, ds,
\]
and consequently for all \( t \in [0,T] \) (4) is true.

Lemma 5. \textit{(Krasnoselskiis fixed point theorem, [16])} Let \( M \) be a closed, bounded, convex and nonempty subset of a Banach space \( X \). Let \( A, B \) be the operators such that
\begin{itemize}
  \item[(a)] \( Ax + By \in M \) whenever \( x, y \in M \);
  \item[(b)]\( A \) is compact and continuous;
  \item[(c)]\( B \) is a contraction mapping.
\end{itemize}
Then there exists \( z \in M \) such that \( z = Az + Bz \).

3. Main results

In this section, the theorems on uniqueness and existence of a solution for boundary value problem (1)-(2) is given. For the forthcoming analysis we impose suitable conditions on the functions involved in boundary value problem (1), (2). We assume the following conditions are set:

(H1) The function \( f : [0,T] \times \mathbb{R}^{n} \to \mathbb{R}^{n} \) is continuous and satisfies the following Lipschitz condition
\[
\| f(t,x) - f(t,y) \| \leq L \| x - y \|, x, y \in \mathbb{R}^{n}, t \in [0,T], L > 0.
\]
(H2) \[
\| f(t,x) \| \leq G, \text{ for all } x \in \mathbb{R}^{n}, t \in [0,T], G \geq 0.
\]

Theorem 2. Assume that \( f : [0,T] \times \mathbb{R}^{n} \to \mathbb{R}^{n} \) is jointly continuous and satisfies (H1) and (H2). If
\[
\left[ \frac{L \| N^{-1} || n \| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{L \| N^{-1} B \| T^{\alpha}}{\Gamma(\alpha + 1)} \right] < 1,
\]
then fractional differential equation (1) with boundary conditions (2) has at least one solution on \( [0,T] \).

Proof. Consider \( B_r = \{ x \in C([0,T];\mathbb{R}^{n}) : \| x \| \leq r \} \), where
\[
r \geq \frac{GT^{\alpha}}{\Gamma(\alpha + 1)} + \| N^{-1} C \| + \frac{G \| n \| \| N^{-1} \| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{G \| N^{-1} B \| T^{\alpha}}{\Gamma(\alpha + 1)}.
\]
Define two mappings \( A_1 \) and \( A_2 \) on \( B_r \) by
\[
(A_1 x)(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - s)^{\alpha-1} f(s, x(s)) \, ds,
\]
(6)
\[(A_2 x)(t) = N^{-1} \left( C - \frac{1}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} f(s, x(s)) \, ds \, dt \right) - \frac{B}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s, x(s)) \, ds \].

(7)

For \( x, y \in B_r \) by (H2), we obtain

\[
\| (A_1 x)(t) + (A_2 y)(t) \| \leq G \frac{\| N \| \| N^{-1} \|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \, ds + \| N^{-1} C \|

+ \frac{G \| n \| \| N^{-1} \|}{\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} \, ds \, dt + \frac{\| N^{-1} B \|}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \, ds

\leq GT^{\alpha} / (\alpha + 1) + \| N^{-1} C \| + \frac{G \| n \| \| N^{-1} \|}{\Gamma(\alpha + 2)} T^{\alpha+1} + \frac{\| N^{-1} B \|}{\Gamma(\alpha + 1)} T^{\alpha} \leq r.

This shows that \( A_1 x + A_2 y \in B_r \). Therefore, condition (a) of Lemma 5 holds. It is claimed that \( A_1 \) is compact and continuous. Continuity of \( f \) implies that the operator (6) is continuous. \( (A_1 x)(t) \) is uniformly bounded on \( B_r \) as

\[
\| A_1 x \| \leq \frac{G T^{\alpha}}{\Gamma(\alpha + 1)}.
\]

Since \( f \) is bounded on the compact set \([0, T] \times B_r\), let \( \sup_{[0,T] \times B_r} \| f(t, x) \| = M_f \). Then, for \( t_1, t_2 \in [0, T], t_1 < t_2 \) we get

\[
\| (A_1 x)(t_2) - (A_1 x)(t_1) \| = \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right) f(s, x(s)) \, ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s)) \, ds

\leq \frac{M_f}{\Gamma(\alpha)} \left( \frac{t_2^{\alpha} - t_1^{\alpha}}{\alpha} \right),
\]

which is independent of \( x \) and tends to zero as \( t_2 \to t_1 \). Therefore, \( A_1 \) is relatively compact on \( B_r \). By Arzela Ascolis Theorem, \( A_1 \) is compact on \( B_r \).

For \( x, y \in B_r \) and \( t \in [0, T] \), by (H1), we have

\[
\| (A_2 x)(t) - (A_2 y)(t) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t-s)^{\alpha-1} (f(s, x(s)) - f(s, y(s))) \, ds \, dt.
\]
\[
\frac{1}{\Gamma(\alpha)} \left\| N^{-1} B \int_0^T (T - s)^{\alpha - 1} (f(s, x(s)) - f(s, y(s))) \, ds \right\| \\
\leq \left[ \frac{L \| N^{-1} \| \|n\| T^{\alpha+1}}{\Gamma(\alpha + 1)} + \frac{L \| N^{-1} B \| T^{\alpha}}{\Gamma(\alpha + 1)} \right] \| x - y \|
\]

It follows from (5) that the operator (7) is a contraction mapping. Thus, by Krasnoselskiis fixed point theorem, (1) - (2) has at least one solution.

**Theorem 3.** Assume that \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function satisfying the assumption \( (H1) \). Then the problems (1) - (2) has a unique solution on \([0, T]\) if

\[ L \Lambda < 1, \]

where \( \Lambda \) is given by

\[ \Lambda = \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\| N^{-1} \| \|n\| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{\| N^{-1} B \| T^\alpha}{\Gamma(\alpha + 1)}. \]

**Proof.** Define a mapping \( F : C([0, T]; \mathbb{R}^n) \to ([0, T]; \mathbb{R}^n) \) by

\[
(Fx)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, x(s)) \, ds \\
+ N^{-1} \left( C - \frac{1}{\Gamma(\alpha)} \int_0^T n(t) \int_0^t (t - s)^{\alpha - 1} f(s, x(s)) \, ds \, dt \right) \\
- \frac{B}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha - 1} f(s, x(s)) \, ds. \quad (8)
\]

Let us first show that \( FB_r \subset B_r \), where is the operator defined by (8) and \( r \geq \frac{M_f L}{1 - L \Lambda} \) with \( M_f = \sup_{t \in [0, T]} |f(t, 0)| \), \( \Lambda = \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\| N^{-1} \| \|n\| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{\| N^{-1} B \| T^\alpha}{\Gamma(\alpha + 1)} \). Then, in view of the assumptions \( (H1) \) and \( (H2) \), we have

\[ |f(t, x)| \leq |f(t, x) - f(t, 0)| + |f(t, 0)| \leq L |x| + M_f \leq L r + M_f. \]

For any \( x \in B_r \), we have

\[ \|Fx\| = \sup_{t \in [0, T]} |Fx(t)| \leq \| N^{-1} C \| \]
\[ (+ (L_r + M_f) \left\{ \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\|N^{-1}\| \|n\| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{\|N^{-1}B\| T^\alpha}{\Gamma(\alpha + 1)} \right\} \leq r, \]

which implies that \( F B_r \subset B_r \). Next, for \( x, y \in C([0, T]; R^n) \) and for each \( t \in [0, T] \), we obtain

\[
\|F(x) - F(y)\| \leq \sup_{[0, T]} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds \\
+ \frac{\|N^{-1}\| \|n\|}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds \\
+ \frac{\|N^{-1}B\|}{\Gamma(\alpha)} \int_0^T (T - s)^{\alpha-1} |f(s, x(s)) - f(s, y(s))| \, ds \\
\leq L \left\{ \frac{T^\alpha}{\Gamma(\alpha + 1)} + \frac{\|N^{-1}\| \|n\| T^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{\|N^{-1}B\| T^\alpha}{\Gamma(\alpha + 1)} \right\} \|x - y\| = L \Lambda \|x - y\|.
\]

Since \( L \Lambda < 1 \) the operator \( F \) is a contraction. By Banach contraction mapping principle the operator \( F \) has a unique fixed point, which means that the problem (1) and (2) has a unique solution for on \([0, T]\).

References


