On the Transformations Preserving Asymptotic Directions of Hypersurfaces in the Euclidean Space

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Abstract. We consider the transformations preserving asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we obtain a system of equations which must be satisfied by transformations.

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1. Introduction

In the Euclidean space, the projective transformation preserves the asymptotic lines of a surface [3]. In [4] the inverse of that problem is considered and it is obtained that the most transformation preserving the asymptotic lines of surfaces in 3-dimensional Euclidean space is the projective one. But that paper has very long

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calculations and it seems very difficult to generalize for the n-dimensional space by using given method. Moreover, since it has some errors that transformation is not the general projective transformation [1].

In this paper, we consider the transformations which preserve the asymptotic directions of hypersurfaces in n-dimensional Euclidean space and we obtain a system of equations. The transformations must satisfy these equations system.

2. The Equation of the Asymptotic Directions of a Hypersurface

In the n-dimensional Euclidean space, a hypersurface can be expressed by the equation

\[ \mathbf{r}(u^1, \ldots, u^{n-1}) = \left( x^1(u^1, \ldots, u^{n-1}), x^2(u^1, \ldots, u^{n-1}), \ldots, x^n(u^1, \ldots, u^{n-1}) \right) \]  

where the metric of the space is given by

\[ ds^2 = (dx^1)^2 + (dx^2)^2 + \ldots + (dx^n)^2. \]

We assume that \( \mathbf{r}(u^1, u^2, \ldots, u^{n-1}) \) is a differentiable function of order 3 and the tangent vectors \( \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{n-1} \) of the hypersurface are linearly independent where

\[ \mathbf{r}_i \equiv \frac{\partial \mathbf{r}}{\partial u^i}, \quad (i = 1, 2, \ldots, n - 1). \]

The first and second fundamental forms of the hypersurface are

\[ I = g_{ij} du^i du^j, \quad II = L_{ij} du^i du^j, \quad (i, j = 1, 2, \ldots, n - 1) \]

where

\[ g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j \]

\[ L_{ij} = \mathbf{r}_{ij} \cdot \mathbf{N}, \quad \left( \mathbf{r}_{ij} \equiv \frac{\partial^2 \mathbf{r}}{\partial u^i \partial u^j} \right). \]
Here $\mathbf{N}$ is the unit normal vector of the hypersurface, that is,

$$\mathbf{r}_i \cdot \mathbf{N} = 0$$  \hspace{1cm} (7)

and

$$\mathbf{N} \cdot \mathbf{N} = 1.$$  \hspace{1cm} (8)

The differential equation of the asymptotic directions of the hypersurface is given by

$$L_{ij} d\mathbf{u}^i d\mathbf{u}^j = 0$$  \hspace{1cm} (9)

[2, p.44 ] and [5, p.134 ].

The system (7) can be written as

$$\mathbf{A} \mathbf{N}^T = 0$$  \hspace{1cm} (10)

where

$$\mathbf{A} = \begin{bmatrix} x^1_1 & x^2_1 & \cdots & x^n_1 \\ x^1_2 & x^2_2 & \cdots & x^n_2 \\ \vdots & \vdots & \ddots & \vdots \\ x^1_{n-1} & x^2_{n-1} & \cdots & x^n_{n-1} \end{bmatrix}, \quad \left( x^k_i = \frac{\partial x^k}{\partial u^i} \right)$$  \hspace{1cm} (11)

and

$$\mathbf{N} = (N_1, N_2, \ldots, N_n).$$  \hspace{1cm} (12)

Since the vectors

$$\mathbf{r}_i = (x^1_i, x^2_i, \ldots, x^n_i), \quad (i = 1, 2, \ldots, n - 1)$$  \hspace{1cm} (13)

are linearly independent, we can assume that

$$\Delta_n = \det \begin{bmatrix} x^1_1 & x^2_1 & \cdots & x^{n-1}_1 \\ x^1_2 & x^2_2 & \cdots & x^{n-1}_2 \\ \vdots & \vdots & \ddots & \vdots \\ x^1_{n-1} & x^2_{n-1} & \cdots & x^{n-1}_{n-1} \end{bmatrix} \neq 0.$$  \hspace{1cm} (14)
Then from (10) and (8) we have

\[ N = \frac{1}{k} \left( \Delta_1, -\Delta_2, \ldots, (-1)^{1+n} \Delta_n \right) \]  

(15)

where \( \Delta_i \) is the determinant of the matrix which is obtained by omitting \( i^{th} \) column in the coefficients matrix \( A \) and

\[ k = \sqrt{\Delta_1^2 + \Delta_2^2 + \ldots + \Delta_n^2}. \]  

(16)

Accordingly, from (6) we get

\[ L_{ij} = \frac{1}{k} [x_{1,ij}^1 \Delta_1 - x_{2,ij}^2 \Delta_2 + \ldots + (-1)^{1+n} x_{ij}^n \Delta_n] \]  

(17)

and so

\[ kL_{ij} = \det \begin{bmatrix} x_{1,ij}^1 & x_{1,1}^1 & x_{1,2}^1 & \cdots & x_{1,n-1}^1 \\ x_{2,ij}^2 & x_{2,1}^2 & x_{2,2}^2 & \cdots & x_{2,n-1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n,ij}^n & x_{n,1}^n & x_{n,2}^n & \cdots & x_{n,n-1}^n \end{bmatrix}, \quad \left( x_{ij}^k = \frac{\partial^2 x^k}{\partial u^i \partial u^j} \right). \]  

(18)

Now for a hypersurface \( S \) let us choose the parameters as

\[ u^1 = x^1, u^2 = x^2, \ldots, u^{n-1} = x^{n-1}. \]  

(19)

Then, the equation of \( S \) becomes

\[ r(x^1, x^2, \ldots, x^{n-1}) = (x^1, x^2, \ldots, x^{n-1}, x^n(x^1, \ldots, x^{n-1})) \]  

(20)

and from (18) we get

\[ kL_{ij} = (-1)^{n+1} x_{ij}^n. \]  

(21)

See also [2, p.36 ].

The differential equation of the asymptotic directions of \( S \), from (9), is obtained as

\[ x_{ij}^n dx^idx^j = 0 \quad (i, j = 1, 2, \ldots, n - 1). \]  

(22)
3. Conditions for a Transformation Preserving the Asymptotic Directions

Here we determine transformations preserving the asymptotic directions of a hypersurface. In the n-dimensional Euclidean space let us consider the coordinate transformation

\[ T : y^a = y^a(x^1, x^2, \ldots, x^n), \quad (a = 1, 2, \ldots, n). \]  

(23)

We assume that \( T \) is differentiable of order 3 and

\[ \Delta = \det \begin{bmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,n} \\ \vdots \\ T_{n,1} & T_{n,2} & \cdots & T_{n,n} \end{bmatrix} \neq 0 \]  

(24)

where

\[ T_{a,b} = \begin{bmatrix} y_{1,b}^1 \\ y_{2,b}^2 \\ \vdots \\ y_{n,b}^n \end{bmatrix}, \quad \left( y_{a,b}^b = \frac{\partial y^a}{\partial x^b}; \; b = 1, 2, \ldots, n \right). \]  

(25)

If the transformation \( T \) is applied to the hypersurface \( S \) which is defined by the equation (20), then we get

\[ T' : y^a = y^a(x^1, x^2, \ldots, x^{n-1}, x^n(x^1, x^2, \ldots, x^{n-1})). \]  

(26)

So the transformation \( T \) transforms the hypersurface \( S \) to a hypersurface \( S^* \) which is given by the equation

\[ r^*(x^1, x^2, \ldots, x^{n-1}) = (y_1^1, y_2^2, \ldots, y_n^n) \]  

(27)

where

\[ y^a = y^a(x^1, x^2, \ldots, x^{n-1}, x^n(x^1, x^2, \ldots, x^{n-1})), \quad (a = 1, \ldots, n). \]

For the hypersurface \( S^* \),

\[ k^* L_{ij}^* = \begin{vmatrix} T_{1,1}' & T_{1,2}' & \cdots & T_{1,n-1}' & T_{1,j}' \\ T_{2,1}' & T_{2,2}' & \cdots & T_{2,n-1}' & T_{2,j}' \\ \vdots \\ T_{n,1}' & T_{n,2}' & \cdots & T_{n,n-1}' & T_{n,j}' \end{vmatrix} \]  

(28)
is obtained from (18), where

\[ T'_{ij} = T_{ij} + T_{ij}x_{ij}^n, \quad T'_{ij} = T_{ij} + T_{ij}x_{ij}^n + T_{ij}x_{ij}^n + T_{ij}x_{ij}^n + T_{ij}x_{ij}^n + T_{ij}x_{ij}^n, \]  

(29)

and

\[
T_{ij} = \begin{bmatrix}
    y_{ij}^1 \\ y_{ij}^2 \\ \vdots \\ y_{ij}^n
\end{bmatrix}, \quad \left( y_{ij}^a = \frac{\partial^2 y^a}{\partial x^i \partial x^j}; i, j = 1, 2, \ldots, n - 1 \right). \tag{30}
\]

Using (29) and (30), from (28) we can write

\[
k^s L_{ii}^s = \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^1_i
\]

\[
+ \left| T_{1,1}T_{n-1,2} \ldots T_{n-1,1}T_{n,n} \right| x^2_i + \ldots + \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ 2 \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ \ldots + 2 \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-2} + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_i
\]

\[
+ \ldots + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-2} + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_i
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-1} + \Delta x^n_{ii} \tag{31}
\]

and

\[
k^s L_{ij}^s = \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ \left| T_{1,1}T_{n-1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_i
\]

\[
+ \ldots + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| x^n_{i-2} + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-1} + 2 \left| T_{1,1}T_{1,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_i
\]

\[
+ \ldots + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-2} + \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_i
\]

\[
+ \left| T_{1,1}T_{2,2} \ldots T_{n-1,1}T_{n,n} \right| (x^n_i)^2 x^n_{i-1} + \Delta x^n_{ij} \tag{32}
\]
The equations (31) and (32) can be written as follows:

\[ + \ldots + \left| T_{1}T_{2} \ldots T_{n}T_{n-1}T_{nj} \right| x_{i}^{n}x_{j}^{n} \]
\[ + \left| T_{1}T_{2} \ldots T_{n-2}T_{nj} \right| x_{i}^{n}x_{j}^{n-1} + \left| T_{1}T_{2} \ldots T_{n-1}T_{in} \right| x_{i}^{n} \]
\[ + \left| T_{n}T_{2} \ldots T_{n-1}T_{in} \right| x_{j}^{n}x_{i}^{n-1} + \left| T_{1}T_{n} \ldots T_{n-1}T_{in} \right| x_{j}^{n}x_{2}^{n} \]
\[ + \ldots + \left| T_{1}T_{2} \ldots T_{n-1}T_{nn} \right| x_{j}^{n}x_{j}^{n-2} \]
\[ + \left| T_{1}T_{2} \ldots T_{n-2}T_{nn} \right| x_{i}^{n}x_{n}^{n-1} + \Delta .x_{ij}^{n}. \quad (32) \]

The differential equation of the asymptotic directions of \( S^{*} \), according to (9), is

\[ L_{ij}^{*}dx^{i}dx^{j} = 0. \quad (33) \]

In order that the transformation \( T \) transforms the asymptotic directions of the hypersurface \( S \) to the asymptotic directions of the hypersurface \( S^{*} \) it must transform the equation (22) to the equation (33). Accordingly, our conditions are

\[ L_{ij}^{*} = tx_{ij}^{n} \quad (34) \]

where \( t \) is an arbitrary function of the variables \( x^{1}, x^{2}, \ldots, x^{n-1} \).

The equations (31) and (32) can be written as follows:

\[ k^{*}L_{ii}^{*} = \Delta_{0}(ii) + \Delta_{1}(ii)x_{i}^{n} + \Delta_{2}(ii)x_{j}^{n} + \ldots + \Delta_{n-1}(ii)x_{n-1}^{n} \]
\[ + 2[\Delta_{0}(in) + \Delta_{1}(in)x_{i}^{n} + \Delta_{2}(in)x_{j}^{n} + \ldots + \Delta_{n-1}(in)x_{n-1}^{n}]x_{i}^{n} \]
\[ + [\Delta_{0}(nn) + \Delta_{1}(nn)x_{i}^{n} + \Delta_{2}(nn)x_{j}^{n} + \ldots + \Delta_{n-1}(nn)x_{n-1}^{n}]x_{j}^{n} \]
\[ + \Delta .x_{ii}^{n}. \quad (35) \]

and

\[ k^{*}L_{ij}^{*} = \Delta_{0}(ij) + \Delta_{1}(ij)x_{i}^{n} + \Delta_{2}(ij)x_{j}^{n} + \ldots + \Delta_{n-1}(ij)x_{n-1}^{n} \]

\[ + 2[\Delta_{0}(in) + \Delta_{1}(in)x_{i}^{n} + \Delta_{2}(in)x_{j}^{n} + \ldots + \Delta_{n-1}(in)x_{n-1}^{n}]x_{j}^{n} \]
\[ + [\Delta_{0}(nn) + \Delta_{1}(nn)x_{i}^{n} + \Delta_{2}(nn)x_{j}^{n} + \ldots + \Delta_{n-1}(nn)x_{n-1}^{n}]x_{i}^{n} \]
\[ + \Delta .x_{ij}^{n}. \quad (36) \]
where $\Delta_0(ab)$ denotes the determinant which is obtained by replacing the $n^{th}$ column with $T_{ab}$ in the determinant $\Delta$ which is defined by (24), and $\Delta_k(ab)$ denotes the determinant which is obtained by replacing the $n^{th}$ column with $T_{ab}$ and $k^{th}$ column with $T_{,n}$ in the determinant $\Delta$. For example,

$$\Delta_2(44) = |T_{1n}T_{3} \ldots T_{n-1}T_{44}|.$$ 

The equations (34) must be satisfied by any hypersurface. So, using the quantities given by (35) in (34) we have the following conditions:

$$\Delta_1(nn) = 0, \Delta_2(nn) = 0, \ldots, \Delta_{n-1}(nn) = 0, \quad (37)$$

$$\Delta_0(ii) = 0, \Delta_1(ii) = 0, \ldots, \Delta_{i-1}(ii) = 0, \Delta_{i+1}(ii) = 0, \ldots, \Delta_{n-1}(ii) = 0, \quad (38)$$

$$\Delta_1(in) = 0, \Delta_2(in) = 0, \ldots, \Delta_{i-1}(in) = 0, \Delta_{i+1}(in) = 0, \ldots, \Delta_{n-1}(in) = 0, \quad (39)$$

$$\Delta_1(ii) + 2\Delta_0(in) = 0, \quad \Delta_0(nn) + 2\Delta_1(in) = 0. \quad (40)$$

From (37) and (38) we respectively get

$$T_{,nn} = 2A_n T_{,n} \quad (41)$$

and

$$T_{,ii} = 2A_i T_{,i} \quad (42)$$

and so

$$T_{,bb} = 2A_b T_{,b} \quad (43)$$
where $A_1, A_2, \ldots, A_n$ are arbitrary functions of variables $x^1, x^2, \ldots, x^n$.

Using (43) in (39) and (40), we have

$$T_{i,n} = A_n T_{i} + A_i T_{n}. \quad (44)$$

Now let us use the quantities given by (36) in (34) which must be satisfied by any hypersurface. Then we have the following conditions:

$$\Delta_0(ij) = 0, \Delta_1(ij) = 0, \ldots, \Delta_{i-1}(ij) = 0, \Delta_{i+1}(ij) = 0,$$

$$\ldots, \Delta_{j-1}(ij) = 0, \Delta_{j+1}(ij) = 0, \ldots, \Delta_{n-1}(ij) = 0$$

$$\Delta_i(ij) + \Delta_0(jn) = 0, \quad \Delta_j(ij) + \Delta_0(in) = 0,$$

$$\Delta_i(in) + \Delta_j(jn) + \Delta_0(nn) = 0 \quad (45)$$

and (37) and (39) again.

From (44) and (45), using (46) we get

$$T_{i,j} = A_j T_{i} + A_i T_{j}. \quad (48)$$

The results (43), (44) and (48) can be expressed by a single equation as

$$T_{a,b} = A_b T_{a} + A_a T_{b}, \quad (a, b = 1, 2, \ldots, n). \quad (49)$$

(47) is automatically satisfied by these results.

From (37) to (40) and from (45) to (47) all equations are satisfied by (49). Thus we have the following theorem.

**Theorem 1.** A transformation $T$ which preserves the asymptotic directions of a hypersurface must satisfy the equations

$$T_{a,b} = A_b T_{a} + A_a T_{b}, \quad (a, b = 1, 2, \ldots, n) \quad (50)$$

where $A_1, A_2, \ldots, A_n$ are arbitrary functions of variables $x^1, x^2, \ldots, x^n$.
References


