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On the Basis Property in $Lp(0,1)$ of the Root Functions of a Class Non Self Adjoint Sturm-Lioville Operators

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Abstract. In the present paper, we prove the basisness of the root functions of the non self adjoint Sturm-Liouville operators with periodic and anti-periodic boundary conditions in space $L_p(0,1), p > 1$. Here we assume that the potential is a complex valued absolutely continuous function in $[0,1]$.

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1. Introduction

Consider the eigenvalue problem for the differential equation

$$\ell(u) \equiv u'' + q(x)u = \lambda u \tag{1}$$

on the interval $(0,1)$ with the periodic

$$u(0) = u(1),\ u'(0) = u'(1), \tag{2}$$

and antiperiodic

$$u(0) = -u(1),\ u'(0) = -u'(1), \tag{3}$$

boundary conditions, where the potential $q(x)$ is an arbitrary complex valued function. In this paper we study the basis property of the root functions of boundary value problems (1),(2) and (1),(3). This problem is important for the study of non selfadjoint Sturm-Liouville operators. It is well known that the basisness of the system root functions of linear differential operators depends on regularity of boundary conditions (in Birkhoff sense strongly regular, see.[1], p.71).

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In 1962 Mikhailov [17], in 1964 Keselman [9] and in 1971 Dunford and Schwartz [4] it was showed the basisness in $L_2(0,1)$ of the root functions of ordinary linear differential operator with regular boundary conditions.

Ionkin [6] in 1976 studied to following boundary value problem

$$u'' + \lambda u = 0,\ u'(0) - u'(1) = 0,\ u(0) = 0$$

whose boundary conditions are regular, but not strongly regular. All the eigenvalues of this problem starting with the second one are double, the general number of associated that the chosen specially system of root functions form an unconditional basis in $L_2(0,1)$.

By Shkalikov [19] in 1979 [see 20], it was proved that the system of root functions of a differential operator with not strongly regular boundary conditions form a Riesz basis with parentheses.

Kerimov and Mamedov [8] in 1998 found conditions on the potential $q(x)$ under which the system eigenfunctions of boundary value problems (1),(2) and (1),(3) forms Riesz basis in $L_2(0,1)$. Namely, they proved the following result: Assume that $q(x) \in C^{(4)}[0,1]$ is complex-valued functions satisfying the condition $q(0) \neq q(1)$, then the root functions of boundary value problems (1),(2) and (1),(3) form Riesz basis in $L_2(0,1)$. The spectral properties of the boundary value problems (1),(2) and (1),(3) were investigated in [14].

Dernek and Veliev [2] in 2005 established conditions in terms of the Fourier coefficients $q_n = (q(x), \exp i2n\pi x)$ of the potential $q(x)$ proved the following result: Assume that the conditions $\lim\limits_{n\to\infty} \frac{\ln|n|}{nq_{2n}} = 0$, $q_{2n} \sim q_{-2n}$ for (1),(2) (and $\lim\limits_{n\to\infty} \frac{\ln|n|}{nq_{2n+1}} = 0$, $q_{2n+1} \sim q_{-2n-1}$ for (1),(3)) hold, where $a_n \sim b_n$ weans that $c_1 \left|b_n\right| < \left|a_n\right| < c_2 \left|b_n\right|$. Then the root functions of the boundary problem (1),(2) (and (1),(3)) form a Riesz basis in $L_2(0,1)$.

Makin [12] in 2005 obtained the following result: Suppose that $q(x) \in W_1^m[0,1]$, $q^{(l)}(0) = q^{(l)}(1)$, for all $l = 0,1,...,m-1$ and $q_{2n} > c_0 n^{-m-1}, 0 < c_1 < \frac{|\alpha_{2n}|}{|\beta_{2n}|} < c_2$ for (1),(2) (and $q_{2n-1} > c_0 n^{-m-1}, 0 < c_1 < \frac{|\alpha_{2n-1}|}{|\beta_{2n-1}|} < c_2, c_0 > 0$ for (1),(3)) for all $n > N_1$, then the root functions of boundary value problem (1),(2) (and (1),(3)) form a Riesz basis in $L_2(0,1)$. Moreover, some theorems for determining whether the root functions form a Riesz basis in $L_2(0,1)$ or not were given in[12]. A classification on the boundary conditions for Sturm-Lionville operator under which the root functions form a Riesz basis in $L_2(0,1)$ is established in [13].

Shkalikov and Veliev [21] in 2008 obtained similar result in terms of the Fourier coefficients q_n on the potential $q(x) \in W_1^p[0,1]$ ($q^{(\ell)}(0) = q^{(\ell)}(1) = 0, 0 \leq \ell \leq s-1$, where $s \leq p$) under which the system of root functions form a Riesz basis in $L_2(0,1)$.

Mamedov and Menken [15] in 2008 showed that the root functions of the boundary-value problems (1),(2) and (1),(3) form a Riesz basis in $L_2(0,1)$ when $q(x) \in C^{(4)}[0,1]$ is complex-valued functions satisfying the condition the conditions $q(0) = q(1)$ and $q'(0) \neq q'(1)$.

In 2008 Djakov and Mitjagin [3] obtained some result on the absence of the Riesz basis property.

In 2010 Kıraç [10] showed that the root functions of the boundary-value problems (1),(2) and (1),(3) form a Riesz basis in $L_2(0,1)$ when $q(x)$ is absolutely continuous function in

$[0,1]$ and $q(0) \neq q(1)$.

By Kurbanov [11] in 2006 it was showed that the root functions of the boundary value problems (1),(2) and (1),(3) forms a basis in $Lp(0,1)$, $p>1$ when $q(x) \in C^{(4)}[0,1]$ and $q(0) \neq q(1)$.

In the case $q(x) \in C^{(4)}[0,1]$, $q(0) = q(1)$ and $q'(0) \neq q'(1)$ Menken and Mamedov [16] in 2010 proved the basisness in $Lp(0,1)$.

The purpose of this paper is to find the weaker conditions on the potential $q(x)$ under which the system of root functions form basis in space $Lp(0,1)$, $p>1$.

The following results plays an important role in the proof of main results.

Theorem 1 ([1, 5]). *The following assertions are equivalent:*

1) *sequence* $\left\{\varphi_j\right\}_{j=1}^{\infty}$ *forms a Riesz basis in* H;

2) *The sequence* $\left\{\varphi_j\right\}_{j=1}^{\infty}$ *is complete in the Hilbert space* H*, there corresponds to it a complete biorthogonal sequence* $\left\{\psi_j\right\}_{j=1}^{\infty}$, *and for any* $f \in H$ *one has* $\sum\limits_{j=1}^{\infty}\left|(f,\varphi_j)\right| < \infty$, $\sum\limits_{j=1}^{\infty}\left|(f,\psi_j)\right|^2 < \infty$.

Theorem 2 ([7]). *A system* $\left\{\varphi_j\right\}_{j=1}^{\infty}$ *is a basis in the Banach space* X *if only if the following conditions are satisfied:*

a) $\left\{\varphi_j\right\}_{j=1}^{\infty}$ *is complete in* X*,*

b) $\left\{\varphi_j\right\}_{j=1}^{\infty}$ *is minimal,*

c) *there exists a number* $M>0$ *such that for each* $f \in X$*, the inequality*

$$\left\|\sum_{j=1}^{N}(f,\psi_j)\varphi_j\right\| \leq M\left\|x\right\|, \ N=1,2,\ldots$$

where the sequence $\left\{\psi_j\right\}_{j=1}^{\infty}$ *is the biorthogonal adjoint system to* $\left\{\varphi_j\right\}_{j=1}^{\infty}$.

Assume that $q(x) \in AC[0,1]$ (absolutely continuous function in $[0,1]$) and $q(0) \neq q(1)$. Under these conditions it is known ([10]) that:

a) all eigenvalues of problem (1),(2) starting from number N are simple and form two infinite sequences.

$$\lambda_{2n} = -(2n\pi)^2 + \frac{q(0)-q(1)}{4n\pi} + o(\frac{1}{n}),$$

$$\lambda_{2n+1} = -(2n\pi)^2 + \frac{q(1)-q(0)}{4n\pi} + o(\frac{1}{n}), \ n=N,N+1,\ldots,$$

and the corresponding eigenfunctions are of the form

$$u_{2n}(x) = \sqrt{2}\sin(2n\pi x - \frac{\pi}{4}) + O(\frac{1}{n}), \tag{4}$$

$$u_{2n+1}(x) = \sqrt{2}\cos(2n\pi x - \frac{\pi}{4}) + O(\frac{1}{n}),\ n = N, N+1, \ldots; \tag{5}$$

b) all eigenvalues of the boundary value problem (1),(3) starting from number N are simple and form two infinite sequences.

$$\lambda_{2n} = -[(2n+1)\pi]^2 + \frac{q(0)-q(1)}{4n\pi} + o(\frac{1}{n}),$$

$$\lambda_{2n+1} = -[(2n+1)\pi]^2 + \frac{q(1)-q(0)}{4n\pi} + o(\frac{1}{n}),\ n = N, N+1, \ldots,$$

and the corresponding eigenfunctions are of the form

$$u_{2n}(x) = \sqrt{2}\sin((2n+1)\pi x - \frac{\pi}{4}) + O(\frac{1}{n}), \tag{6}$$

$$u_{2n+1}(x) = \sqrt{2}\cos((2n+1)\pi x - \frac{\pi}{4}) + O(\frac{1}{n}),\ n = N, N+1, \ldots. \tag{7}$$

In the present paper, in Section 2 we using the asymptotic formulas (4)-(7) of eigenfunctions of the boundary problems (1), (2) and Theorem 1 shown Riesz basisness in $L_2(0,1)$. In Section 3, using Theorem 2 and F.Riesz Theorem (see [22], p.154), we prove the basisness in $L_p(0,1)$ of the root functions of the periodic and anti-periodic boundary value problems.

2. The Riesz Basisness in $L_2(0,1)$

Firstly, by a different method, using the asymptotic formulas (4)-(7) and the Theorem 1 we will show the Riesz basisness in $L_2(0,1)$ of the root functions of the boundary problems (1),(2) and (1),(3).

Theorem 3. *Suppose that $q(x) \in AC[0,1]$ and $q(0) \neq q(1)$. Then the system of root functions of the boundary value problems (1),(2) and (1),(3) forms a Riesz basis in $L_2(0,1)$.*

Proof. It is well known that the system of eigenfunctions and associated eigenfunctions of problem (1),(2) is complete in $L_2(0,1)$. The system of the root functions is minimal in $L_2(0,1)$. The minimality of this system follows from the fact that this system has a biorthogonal system consisting of the root functions of the adjoint operator

$$\begin{aligned} l^*(v) &= v'' + \overline{q(x)}v, \\ v(1) &= v(0),\ v'(1) = v'(0). \end{aligned}$$

For any $f \in L_2(0,1)$, with a direct computation we have that $\sum\limits_{n=N}^{\infty} \left|(f,u_{2n})\right|^2 < \infty$, $\sum\limits_{n=N}^{\infty} \left|(f,u_{2n+1})\right|^2 < \infty$. On the other hand, the eigenfunctions of the adjoint operator have of the form

$$\upsilon_{2n}(x) = \sqrt{2}\sin((2n+1)\pi x - \frac{\pi}{4}) + O(\frac{1}{n}), \tag{8}$$

$$\upsilon_{2n+1}(x) = \sqrt{2}\cos((2n+1)\pi x - \frac{\pi}{4}) + O(\frac{1}{n}),\ n = N, N+1, \ldots. \tag{9}$$

and the inequalities $\sum\limits_{n=N}^{\infty} \left|(f,\upsilon_{2n})\right|^2 < \infty$ and $\sum\limits_{n=N}^{\infty} \left|(f,\upsilon_{2n+1})\right|^2 < \infty$ hold. According to Theorem 1, the root functions of the boundary problem (1),(2) form a Riesz basis in $L_2(0,1)$.

Similarly, is proved the basisness in $L_2(0,1)$ of root functions of boundary problem (1),(3). This completes the proof.

3. The Basisness in $Lp\,(0,1)$, $p > 1$

Theorem 4. *Suppose that $q(x) \in AC[0,1]$, $q(0) \neq q(1)$. Then the system of the root functions of the boundary problems (1),(2) and (1),(3) forms a basis in the space $L_p(0,1)$ $(1 < p < \infty)$.*

Proof. Let $\psi_1(x) = 1$, $\psi_{2n}(x) = \sqrt{2}\sin(2n\pi x - \frac{\pi}{4})$, $\psi_{2n+1}(x) = \sqrt{2}\cos(2n\pi x - \frac{\pi}{4}), n = 1,2,\ldots$. The systems $\{\psi_n(x)\}_{n=1}^{\infty} = \{\psi_1(x), \psi_{2n}(x), \psi_{2n+1}(x)\}_{n=1}^{\infty}$ is a basis in the space $L_p\,(0,1)$. Moreover, if $p = 2$, this basis is orthonormal.

We introduce notation:

$$\{u_n(x)\}_{n=1}^{\infty} = \{u_1(x), u_{2n}(x), u_{2n+1}(x)\}_{n=1}^{\infty},$$

$$\{\upsilon_n(x)\}_{n=1}^{\infty} = \{\upsilon_1(x), \upsilon_{2n}(x), \upsilon_{2n+1}(x)\}_{n=1}^{\infty}.$$

It follows from the asymptotic formulas (4),(5) and (8),(9) that

$$u_n(x) = \psi_n(x) + O(\frac{1}{n}),\ \upsilon_n(x) = \psi_n(x) + O(\frac{1}{n}) \tag{10}$$

for sufficiently large n.

Let $1 < p < 2$. By Theorem 3, the system of the root functions $\{u_n(x)\}_{n=1}^{\infty}$ is basis in $L_2\,(0,1)$ and $\{\upsilon_n(x)\}_{n=1}^{\infty}$ is biorthogonal to the system $\{u_n(x)\}_{n=1}^{\infty}$. Thus, this system is complete in $L_p\,(0,1)$. Therefore, by Theorem 2, to prove the basis property of $\{u_n(x)\}_{n=1}^{\infty}$ in $Lp\,(0,1)$, it is necessary and sufficient to prove that existence of a constant $M > 0$ such that

$$\left\|\sum_{n=1}^{N}(f,\upsilon_n)u_n\right\|_p \leq M\left\|f\right\|_p,\ N = 1,2,\ldots, \tag{11}$$

for all $f \in L_p(0,1)$, where $\|\cdot\|_p$ denotes the norm in $L_p(0,1)$.

By (10) we obtain

$$\begin{aligned}\left\|\sum_{n=1}^{N}(f,\upsilon_n)u_n\right\|_p &\leq \left\|\sum_{n=1}^{N}(f,\psi_n)\psi_n\right\|_p+\left\|\sum_{n=1}^{N}(f,\psi_n)O(\frac{1}{n})\right\|_p+\\ &\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))\psi_n\right\|_p+\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))O(\frac{1}{n})\right\|_p. \qquad (12)\end{aligned}$$

We shall now prove that all the summands on the right side of (12) are bounded from above by constant $\|f\|_p$.

Since the system $\{\psi_n(x)\}_{n=1}^{\infty}$ is a basis in the space $L_p(0,1)$, applying Theorem 2, we have

$$\left\|\sum_{n=1}^{N}(f,\psi_n)\psi_n\right\|_p \leq C\left\|f\right\|_p,\ C=const\ (N=1,2,\cdots) \qquad (13)$$

for arbitrary for all $f\in L_p(0,1)$.

By F.Riesz Theorem (see [22], p.154) for an arbitrary function $f\in L_p(0,1)$ one has the estimate

$$\begin{aligned}\left\|\sum_{n=1}^{N}(f,\psi_n)O(\frac{1}{n})\right\|_p &\leq C\sum_{n=1}^{N}(f,\psi_n)\frac{1}{n}\\ &\leq C\left(\sum_{n=1}^{N}\left|(f,\psi_n)\right|^q\right)^{\frac{1}{q}}\cdot\left(\sum_{n=1}^{N}\frac{1}{n^p}\right)^{\frac{1}{p}}\leq C\left\|f\right\|_p, \qquad (14)\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Using the Parseval's equality we have

$$\begin{aligned}\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))\psi_n\right\|_p &\leq \left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))\psi_n\right\|_2=\left(\sum_{n=1}^{N}\left|(f,O(\frac{1}{n}))\right|^2\right)^{\frac{1}{2}}\\ &\leq C\left\|f\right\|_1\left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}}\leq C\left\|f\right\|_p, \qquad (15)\end{aligned}$$

$$\left\|\sum_{n=1}^{N}(f,O(\frac{1}{n}))O(\frac{1}{n})\right\|_p\leq C\left\|f\right\|_1\left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}}\leq C\left\|f\right\|_p. \qquad (16)$$

Using the inequalities (13)-(16) in the estimate (12) we have (11). Thus the proof of assertion (11) is complete. Consequently, the system $\{u_n(x)\}_{n=1}^{\infty}$ is a basis in the space $Lp(0,1)$, $1<p<2$.

Now assume that $2<p<\infty$. It is clear that the system $\{u_n(x)\}_{n=1}^{\infty}$ is a basis in $Lq(0,1)$. By Corollary 2 in [7] (in Section I) it follows that the system $\{\upsilon_n(x)\}_{n=1}^{\infty}$ is a basis in the space $L_p(0,1)$, where $\frac{1}{p}+\frac{1}{q}=1$. Note that $1<q<2$. Using the discussions introduced above entirely analogously it is proved that the system $\{\upsilon_n(x)\}_{n=1}^{\infty}$ is a basis in $Lq(0,1)$. It follows that $\{u_n(x)\}_{n=1}^{\infty}$ is a basis in the space $Lp(0,1)$, $p>2$.

Entirely analogously it is not difficult to prove that the system of the root functions of the boundary problem (1),(3) form a basis in the space $L_p(0,1)$ $(1<p<\infty)$. The proof of the theorem is complete.

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