Random Stability of a Functional Equation Related to an Inner Product Space

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Abstract. In [14], Th.M. Rassias introduced the following equality

$$\sum_{i,j=1}^{n} \|x_i - x_j\|^2 = 2n \sum_{i=1}^{n} \|x_i\|^2, \quad \sum_{i=1}^{n} x_i = 0$$

for a fixed integer $n \geq 3$. For a mapping $f : X \rightarrow Y$, where $X$ is a vector space and $Y$ is a complete random normed space, we consider the following functional equation

$$\sum_{i,j=1}^{n} f(x_i - x_j) = 2n \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \ldots, x_n \in X$ with $\sum_{i=1}^{n} x_i = 0$. In this paper, we prove the Hyers-Ulam stability of the functional equation (1) related to an inner product space.

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1. Introduction

A square norm on an inner product space satisfies the parallelogram equality

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

From the above equation, we consider the following functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

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A square norm on an inner product space also satisfies

\[
\sum_{i,j=1}^{3} \|x_i - x_j\|^2 = 6 \sum_{i=1}^{3} \|x_i\|^2
\]

for all \(x_1, x_2, x_3 \in \mathbb{R}\) with \(x_1 + x_2 + x_3 = 0\) [see 14]. From the above equality we can define the functional equation

\[
f(x - y) + f(2x + y) + f(x + 2y) = 3f(x) + 3f(y) + 3f(x + y),
\]

which is called a quadratic functional equation. In fact, \(f(x) = ax^2\) in \(\mathbb{R}\) satisfies the quadratic functional equation.

The aim of this paper is to investigate the Hyers-Ulam stability of additive-quadratic functional equation in a random normed space related to an inner product space.

Throughout this paper, we use the definition of a random normed space as in \([1, 10, 15, 16]\). \(\Delta^+\) is the space of distribution functions that is, the space of all mappings \(F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]\) which is left-continuous and non-decreasing on \(\mathbb{R}\), \(F(0) = 0\) and \(F(+\infty) = 1\). \(D^+\) is a subset of \(\Delta^+\) consisting of all functions \(F\) for which \(\lim_{t \to \infty} F(t) = 1\), where \(l^-F(x)\) denotes the left limit of the function \(F\) at the point \(x\). The space \(\Delta^+\) is partially ordered by the usual point-wise ordering of functions. The maximal element for \(\Delta^+\) in this order is the distribution function \(\varepsilon_0\) given by

\[
\varepsilon_0(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
1, & \text{if } t > 0.
\end{cases}
\]

**Definition 1** ([15]). A mapping \(T : [0,1] \times [0,1] \to [0,1]\) is a continuous triangular norm (briefly, a continuous t-norm) if \(T\) satisfies the following conditions:

(a) \(T\) is commutative and associative;
(b) \(T\) is continuous;
(c) \(T(a, 1) = a\) for all \(a \in [0,1]\);
(d) \(T(a, b) \leq T(c, d)\) whenever \(a \leq c\) and \(b \leq d\) for all \(a, b, c, d \in [0,1]\).

Recall that if \(T\) is a t-norm and \(\{x_n\}\) is a sequence of numbers in \([0,1]\), then \(T^n_{i=1} x_i\) is defined recurrently by \(T^1_{i=1} x_i = x_1\) and \(T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n)\) for \(n \geq 2\) [see 3]. \(T^\infty_{i=1} x_i\) is defined as \(\lim_{m \to \infty} T^m_{i=1} x_i\).
Definition 2 ([16]). A random normed space (briefly, RN-space) is a triple \((X, \mu, T)\), where \(X\) is a vector space, \(T\) is a continuous \(t\)-norm and \(\mu\) is a mapping from \(X\) into \(D^+\) satisfies the following conditions:

(RN1) \(\mu_x(t) = \varepsilon_0(t)\) for all \(t > 0\) if and only if \(x = 0\);

(RN2) \(\mu_{ax}(t) = \mu_x\left(\frac{t}{|a|}\right)\) for all \(x \in X\), \(a \neq 0\);

(RN3) \(\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))\) for all \(x, y \in X\) and \(t, s \geq 0\).

A sequence \(\{x_n\}\) in an RN-space \((X, \mu, T)\) is said to be convergent to \(x\) in \(X\) if, for every \(\varepsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N\) such that \(\mu_{x_n-x}(\varepsilon) > 1 - \lambda\) whenever \(n \geq N\). An RN-space \((X, \mu, T)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent to a point in \(X\).

The Hyers-Ulam stability of functional equations in random normed spaces and fuzzy normed spaces has been studied [see 3, 4, 6, 8, 9, 11, 12]. Let \(V, W\) be vector spaces. It is shown that if a mapping \(f : V \rightarrow W\) satisfies the functional equation (1), then the mapping \(f\) is the sum of an additive mapping and a quadratic mapping [see 2]. In this paper, we investigate the Hyers-Ulam stability of the functional equation (1) in RN-spaces.

Throughout this paper, assume that \(X\) is a vector space and that \((Y, \mu, T)\) is a complete RN-space.


We investigate the functional equation (1) for an odd mapping in RN-spaces.

For a given mapping \(f : X \rightarrow Y\), we define

\[
Df(x_1, \ldots, x_n) := \sum_{i,j=1}^{n} f(x_i - x_j) - 2n \sum_{i=1}^{n} f(x_i)
\]

for all \(x_1, \ldots, x_n \in X\) with \(\sum_{i=1}^{n} x_i = 0\).

For an odd mapping \(f : X \rightarrow Y\), we note that if \(f\) satisfies

\[
Df(x_1, x_2, \ldots, x_n) = 0
\]

for all \(x_1, \ldots, x_n \in X\) with \(\sum_{i=1}^{n} x_i = 0\) then the mapping \(f\) is additive.

We prove the Hyers-Ulam stability of the functional equation (1) of an odd mapping in RN-spaces.

Theorem 1. Let \(f : X \rightarrow Y\) be an odd mapping for which there is a \(\rho : X^n \rightarrow D^+\) \((\rho(x_1, x_2, \ldots, x_n))\) is denoted by \(\rho(x_1, x_2, \ldots, x_n)\) such that

\[
\mu_{Df(x_1, x_2, \ldots, x_n)}(t) \geq \rho(x_1, x_2, \ldots, x_n)(t)
\]

for all \((x_1, x_2, \ldots, x_n) \in X^n\) and all \(t > 0\). If

\[
T_{k=1}^{\infty} \rho\left(\frac{t}{2^{2k+1}}, \frac{t}{2^{2k+1}}, \ldots, \frac{t}{2^{2k+1}}, 0, \ldots, 0\right) \left(\frac{nt}{2^{2k+l-2}}\right) = 1
\]

(3)
for all \( x, y \in X, \) all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \), then there exists a unique additive mapping \( A : X \rightarrow Y \) such that

\[
\mu_{f(x)-A(x)}(t) \geq T_k^{\infty} \rho \left( \frac{t}{2^{2k-2}} \right) \left( \frac{nt}{2^{2k-2}} \right)
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. Putting \( x_1 = x_2 = \frac{x}{2^k}, x_3 = -x, x_4 = \ldots = x_n = 0 \) in (2), we get

\[
\mu_{2^k \cdot f(x)-2^k f \left( \frac{x}{2^k} \right)}(t) \geq \rho \left( \frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0 \right) (2nt)
\]

which is equivalent to

\[
\mu_{2^k \cdot f(x)-2^k f \left( \frac{x}{2^k} \right)}(t) \geq \rho \left( \frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0 \right) (2nt)
\]

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) and \( t \) by \( \frac{x}{2^k} \) and \( \frac{t}{2^k} \), respectively in the above inequality, we get

\[
m_{2^k \cdot f(x)-2^k f \left( \frac{x}{2^k} \right)} \left( \frac{t}{2^k} \right) \geq \rho \left( \frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0 \right) \left( \frac{nt}{2^{2k-2}} \right)
\]

for all \( x \in X \) and all \( t > 0 \).

Since \( \mu_x(s) \leq \mu_x(t) \) for all \( s \) and \( t \) with \( 0 < s \leq t \), we obtain

\[
m_{f(x)-2^m f \left( \frac{x}{2^m} \right)}(t) \geq \mu_{2^m} \left( 2^k \cdot f \left( \frac{x}{2^k} \right)-2^k f \left( \frac{x}{2^k} \right) \right) \left( \frac{t}{2^k} \right)
\]

\[
\geq \mu_{2^m} \left( 2^k \cdot f \left( \frac{x}{2^k} \right)-2^k f \left( \frac{x}{2^k} \right) \right) \left( \sum_{k=1}^{m} \frac{t}{2^k} \right)
\]

\[
\geq T_k^{m} \rho \left( \frac{x}{2^k}, -\frac{x}{2^k}, 0, \ldots, 0 \right) \left( \frac{nt}{2^{2k-2}} \right)
\]

Replacing \( x \) by \( \frac{x}{2^m} \) in the above inequality, we get

\[
m_{f \left( \frac{x}{2^m} \right)-2^m f \left( \frac{x}{2^m} + \frac{x}{2^{m+l}} \right)}(t) \geq T_k^{m} \rho \left( \frac{x}{2^{2k+l}}, -\frac{x}{2^{2k+l}}, 0, \ldots, 0 \right) \left( \frac{nt}{2^{2k+l-2}} \right)
\]

which is equivalent to

\[
m_{2^l f \left( \frac{x}{2^m} \right)-2^m f \left( \frac{x}{2^m + \frac{x}{2^{m+l}} \right)}(t) \geq T_k^{m} \rho \left( \frac{x}{2^{2k+l}}, -\frac{x}{2^{2k+l}}, 0, \ldots, 0 \right) \left( \frac{nt}{2^{2k+l-2}} \right)
\]

for all \( x \in X, \) all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \).
Since the right hand side of the inequality (6) tends to 1 as $m \to \infty$ by (3), the sequence \( \{2^m f \left( \frac{x}{2^m} \right) \} \) is a Cauchy sequence. Thus we define $A(x) := \lim_{m \to \infty} 2^m f \left( \frac{x}{2^m} \right)$ for all $x \in X$, which is an odd mapping.

Now we show that $A$ is an additive mapping. By (2), we get

$$\mu_{2^m}(f \left( \frac{x+y}{2^m} \right) - f \left( \frac{x}{2^m} \right) - f \left( \frac{y}{2^m} \right)) \geq \rho \left( \left( \frac{x}{2^m}, \frac{y}{2^m}, \ldots, \frac{x+y}{2^m} \right), 0, \ldots, 0 \right) \left( \frac{nt}{2^{m-1}} \right).$$

Taking the limit as $m \to \infty$ in the above inequality, by (4), the mapping $A$ is additive. By letting $l = 0$ and taking the limit as $m \to \infty$ in (6), we get (5).

Finally, to prove the uniqueness of the additive mapping $A$ subject to (5), let us assume that there exists another additive mapping $B$ which satisfies (5). Since

$$\mu_{A(x) - B(x)}(2t) = \mu_{A(x) - 2^m f \left( \frac{x}{2^m} \right) + 2^m f \left( \frac{x}{2^m} \right) - B(x)}(2t) \geq T \left( \mu_{A(x) - 2^m f \left( \frac{x}{2^m} \right)}(t), \mu_{2^m f \left( \frac{x}{2^m} \right) - B(x)}(t) \right)$$

and

$$\lim_{m \to \infty} \mu_{A(x) - 2^m f \left( \frac{x}{2^m} \right)} = \lim_{m \to \infty} \mu_{B(x) - 2^m f \left( \frac{x}{2^m} \right)} = 1$$

for all $x \in X$ and all $t > 0$, we get

$$\lim_{m \to \infty} T \left( \mu_{A(x) - 2^m f \left( \frac{x}{2^m} \right)}(t), \mu_{2^m f \left( \frac{x}{2^m} \right) - B(x)}(t) \right) = 1.$$

Thus we have $A = B$.

**Corollary 1.** Let $\theta \geq 0$ and let $p$ be a constant with $p > 1$. For a normed vector space $X$ and complete RN-space $Y$, let $f : X \to Y$ be an odd mapping satisfying

$$\mu_{Df(x_1, x_2, \ldots, x_n)}(t) \geq \frac{t}{t + \theta \sum_{i=1}^n \|x_i\|^p}$$

for all $(x_1, x_2, \ldots, x_n) \in X$ with $\sum_{i=1}^n x_i = 0$ and all $t > 0$. If

$$T_k^{\infty} \left( \frac{2^{(k+1)p} nt}{2^{(k+1)p} nt + 2^{2k+1-2}(2 + 2^p) \theta \|x\|^p} \right) = 1$$

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\mu_{f(x) - A(x)}(t) \geq T_k^{\infty} \left( \frac{2^{kp} nt}{2^{kp} nt + 2^{2k-2}(2 + 2^p) \theta \|x\|^p} \right)$$

for all $x \in X$ and all $t > 0$. 

Proof. If we define

\[ P(x_1, x_2, \ldots, x_n)(t) = \frac{t}{t + \theta \sum_{i=1}^{n} ||x_i||^p} \]

and apply Theorem 1, then we get the desired result.

**Theorem 2.** Let \( f : X \to Y \) be an odd mapping for which there is a \( \rho : X^n \to D^+ \) satisfying (2). If

\[ T_{k=1}^{\infty} \rho(\sum_{l=0}^{2^k-2} x, 2^{k-l-1} x, -2^{k-l-1} x, 0, \ldots, 0)(2^{l+1} nt) = 1 \]  \hspace{1cm} (7)

and

\[ \lim_{m \to \infty} \rho(\sum_{l=0}^{2^m} x, 2^{m-l} y, -2^m x + y, 0, \ldots, 0)(2^{m+1} nt) = 1 \]  \hspace{1cm} (8)

for all \( x, y \in X, \) all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \), then there exists a unique additive mapping \( A : X \to Y \) such that

\[ \mu_{f(x) - A(x)}(t) \geq T_{k=1}^{\infty} \rho(\sum_{l=0}^{2^k-2} x, 2^{k-l-1} x, -2^{k-l-1} x, 0, \ldots, 0)(2^{l+1} nt) \]  \hspace{1cm} (9)

for all \( x \in X \) and all \( t > 0 \).

Proof. Putting \( x_1 = x_2 = x, x_3 = -2x, x_4 = \ldots = x_n = 0 \) in (2), we get

\[ \mu_{2n(f(2x) - 2f(x))}(t) \geq \rho(x, x, -2x, 0, \ldots, 0)(t) \]

which is equivalent to

\[ \mu_{f(x) - \frac{1}{2} f(2x)}(t) \geq \rho\left(\frac{1}{2}, \frac{1}{2}, -x, 0, \ldots, 0\right)(4nt) \]

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) and \( t \) by \( 2^{k-1} x \) and \( 2t \), respectively, in the above inequality, we get

\[ \mu_{\frac{1}{2^{k-1}} f(2^{k-1} x) - \frac{1}{2} f(2^k x)}\left(\frac{t}{2^k}\right) \geq \rho(2^{k-2} x, 2^{k-2} x, -2^{k-1} x, 0, \ldots, 0)(2nt) \]

for all \( x \in X \) and all \( t > 0 \).

Since \( \mu_x(s) \leq \mu_x(t) \) for all \( s \) and \( t \) with \( 0 < s \leq t \), we obtain

\[ \mu_{f(x) - \frac{1}{2} f(2^m x)}(t) \leq \sum_{k=1}^{m} \left(\frac{1}{2^{k-1}} f(2^{k-1} x) - \frac{1}{2} f(2^k x)\right)(t) \]

\[ \geq \mu_{\sum_{k=1}^{m} \left(\frac{1}{2^{k-1}} f(2^{k-1} x) - \frac{1}{2} f(2^k x)\right) \left(\sum_{k=1}^{m} \frac{t}{2^k}\right) \geq T_{k=1}^{m} \rho(2^{k-2} x, 2^{k-2} x, -2^{k-1} x, 0, \ldots, 0)(2nt) \]

Replacing \( x \) by \( 2^k x \) in the above inequality, we get

\[ \mu_{f(2^k x) - \frac{1}{2} f(2^{k+m} x)}(t) \geq T_{k=1}^{m} \rho(2^{k+l-2} x, 2^{k+l-2} x, -2^{k+l-1} x, 0, \ldots, 0)(2nt) \]
which is equivalent to
\[
\mu \frac{1}{2^m} f(2^m x) - \frac{1}{2^{m+1}} f(2^{m+1} x)(t) \geq T_{k=1}^m \rho(2^{k+1-2} x, 2^{k+1-2} x, -2^{k+1-1} x, 0, \ldots, 0) \left(2^{l+1} nt\right)
\]
for all \(x \in X\), all \(t > 0\) and all \(l = 0, 1, 2, \ldots\).

Since the right hand side of the inequality (10) tends to 1 as \(m \to \infty\) by (7), the sequence \(\left\{\frac{1}{2^m} f(2^m x)\right\}\) is a Cauchy sequence. Thus we define \(A(x) := \lim_{m \to \infty} \frac{1}{2^m} f(2^m x)\) for all \(x \in X\), which is an odd mapping.

Now we show that \(A\) is an additive mapping. By (2), we get
\[
\mu \frac{1}{2^m} (f(2^m(x+y)) - f(2^m x) - f(2^m y))(t) \geq \rho(2^m x, 2^m y, -2^m(x+y), 0, \ldots, 0) (2^{m+1} nt).
\]
Taking the limit as \(m \to \infty\) in the above inequality, by (8) the mapping \(A\) is additive. By letting \(l = 0\) and taking the limit as \(m \to \infty\) in (10), we get (9).

The rest of the proof is the same as in the proof of Theorem 1.

**Corollary 2.** Let \(\theta \geq 0\) and let \(p\) be a constant with \(0 < p < 1\). For a normed vector space \(X\) and complete RN-space \(Y\), let \(f : X \to Y\) be an odd mapping satisfying
\[
\mu_D f(x_1, x_2, \ldots, x_n)(t) \geq \frac{t}{t + \theta \sum_{i=1}^n \|x_i\|^p}
\]
for all \((x_1, x_2, \ldots, x_n) \in X\) with \(\sum_{i=1}^n x_i = 0\) and all \(t > 0\). If
\[
T_{k=1}^\infty \left(\frac{2^{l+1} nt}{2^{l+1} nt + 2^{(k+1-1)p}(2^{1-p} + 1)\theta \|x\|^p}\right) = 1
\]
for all \(x \in X\), all \(t > 0\) and all \(l = 0, 1, 2, \ldots\), then there exists a unique additive mapping \(A : X \to Y\) such that
\[
\mu f(x) - A(x)(t) \geq T_{k=1}^\infty \left(\frac{2nt}{2nt + 2^{(k-1)p}(2^{1-p} + 1)\theta \|x\|^p}\right)
\]
for all \(x \in X\) and all \(t > 0\).

**Proof.** If we define
\[
\rho(x_1, x_2, \ldots, x_p)(t) = \frac{t}{t + \theta \sum_{i=1}^n \|x_i\|^p}
\]
and apply Theorem 2, then we get the desired result.

We prove the Hyers-Ulam stability of the functional equation (1) of an even mapping in RN-spaces.

For an even mapping \( f : X \to Y \) with \( f(0) = 0 \), we note that if \( f \) satisfies
\[
Df(x_1, x_2, \ldots, x_n) = 0
\]
for all \( x_1, \ldots, x_n \in X \) with \( \sum_{i=1}^{n} x_i = 0 \) then the mapping \( f \) is quadratic.

**Theorem 3.** Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \) for which there is a \( \rho : X^n \to D^+ \) satisfying (2). If
\[
T_{k=1}^{(n)} \rho \left( \frac{1}{2^{3k+2l-3}}, \ldots, 0 \right) \left( \frac{t}{2^{3k+2l-3}} \right) = 1
\]
and
\[
\lim_{m \to \infty} \rho \left( \frac{1}{2^{3m}}, \frac{x+y}{2^{3m}}, 0, \ldots, 0 \right) \left( \frac{t}{2^{3m}} \right) = 1
\]
for all \( x, y \in X \), all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \), then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\mu_{f(x) - Q(x)}(t) \geq T_{k=1}^{(n)} \rho \left( \frac{1}{2^{3k+2l-3}}, \ldots, 0 \right) \left( \frac{t}{2^{3k+2l-3}} \right)
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** Putting \( x_1 = x, x_2 = -x, x_3 = \ldots = x_n = 0 \) in (2), we get
\[
\mu_{2(f(2x) - 4f(x))}(t) \geq \rho(x, -x, 0, \ldots, 0)(t)
\]
which is equivalent to
\[
\mu_{f(x) - 4f\left( \frac{x}{2} \right)}(t) \geq \rho(x, -x, 0, \ldots, 0)(2t)
\]
for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) and \( t \) by \( \frac{x}{2^{3k-1}} \) and \( \frac{t}{2^{3k-1}} \), respectively in the above inequality, we get
\[
\mu_{4^{k-1}f\left( \frac{x}{2^{3k-1}} \right) - 4^k f\left( \frac{x}{2^k} \right)} \left( \frac{t}{2^k} \right) \geq \rho\left( \frac{x}{2^{3k-1}}, -\frac{x}{2^{3k-1}}, 0, \ldots, 0 \right) \left( \frac{t}{2^{3k-1}} \right)
\]
for all \( x \in X \) and all \( t > 0 \).

Since \( \mu_{x}(s) \leq \mu_{x}(t) \) for all \( s \) and \( t \) with \( 0 < s \leq t \), we obtain
\[
\mu_{f(x) - 4^m f\left( \frac{x}{2^m} \right)}(t) = \mu_{\sum_{k=1}^{m} \left( 4^{k-1} f\left( \frac{x}{2^{3k-1}} \right) - 4^k f\left( \frac{x}{2^k} \right) \right)}(t)
\]
\[
\geq \mu_{\sum_{k=1}^{m} \left( 4^{k-1} f\left( \frac{x}{2^{3k-1}} \right) - 4^k f\left( \frac{x}{2^k} \right) \right)} \left( \sum_{k=1}^{m} \frac{t}{2^k} \right)
\]
\[
\geq T_{k=1}^{(m)} \rho \left( \frac{1}{2^{3m}}, -\frac{1}{2^{3m}}, 0, \ldots, 0 \right) \left( \frac{t}{2^{3m}} \right)
\]
Replacing $x$ by $\frac{x}{2^m}$ in the above inequality, we get

\[
\mu_f\left(\frac{x}{2^m}\right) - 4^m f\left(\frac{x}{2^m}\right)(t) \geq T_{k=1}^{m} \mu_\theta \left(\frac{\frac{x}{2^m}}{2^{k-1}}, \ldots, 0, \ldots, 0\right) \left(\frac{t}{2^{3k-3}}\right)
\]

which is equivalent to

\[
\mu_{4^m} f\left(\frac{x}{2^m}\right) - 4^m f\left(\frac{x}{2^m}\right)(t) \geq T_{k=1}^{m} \mu_\theta \left(\frac{\frac{x}{2^m}}{2^{k-1}}, \ldots, 0, \ldots, 0\right) \left(\frac{t}{2^{3k+2l-3}}\right)
\]

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$.

Since the right hand side of the inequality (14) tends to 1 as $m \to \infty$ by (11), the sequence $\{4^m f\left(\frac{x}{2^m}\right)\}$ is a Cauchy sequence. Thus we define $Q(x) := \lim_{m \to \infty} 4^m f\left(\frac{x}{2^m}\right)$ for all $x \in X$, which is an even mapping.

Now we show that $Q$ is a quadratic mapping. By (2), we get

\[
\mu_{4^m} \left(f\left(\frac{x}{2^m}\right) + f\left(\frac{2x+y}{2^m}\right) + f\left(\frac{x+2y}{2^m}\right) - 3 f\left(\frac{x+y}{2^m}\right) - 3 f\left(\frac{x}{2^m}\right) - 3 f\left(\frac{y}{2^m}\right)\right)(t)
\]

\[
\geq \mu_\theta \left(\frac{\frac{x}{2^m}, \frac{x+y}{2^m}, \frac{x+2y}{2^m}, 0, \ldots, 0}{2^{2m-1}}\right) \left(\frac{t}{2^{2m-1}}\right).
\]

Taking the limit as $m \to \infty$ in the above inequality, by (12), the mapping $Q$ is quadratic. Moreover, letting $l = 0$ and taking the limit as $m \to \infty$ in (14), we get (13).

The rest of the proof is the same as in the proof of Theorem 1.

**Corollary 3.** Let $\theta \geq 0$ and let $p$ be a constant with $p > 2$. For a normed vector space $X$ and complete RN-space $Y$, let $f : X \to Y$ be an even mapping satisfying

\[
\mu_{Df(x_1, x_2, \ldots, x_n)}(t) \geq \frac{t}{t + \theta \sum_{i=1}^{n} ||x_i||^p}
\]

for all $(x_1, x_2, \ldots, x_n) \in X$ with $\sum_{i=1}^{n} x_i = 0$ and all $t > 0$. If

\[
T_{k=1}^{\infty} \left(\frac{2^{(k+1)p} t}{2^{(k+1)p} t + 2^{3k+2l-2} \theta ||x||^p}\right) = 1
\]

for all $x \in X$, all $t > 0$ and all $l = 0, 1, 2, \ldots$, then there exists a unique quadratic mapping $Q : X \to Y$ such that

\[
\mu_{f(x) - Q(x)}(t) \geq T_{k=1}^{\infty} \left(\frac{2^{kp} t}{2^{kp} t + 2^{3k-2} \theta ||x||^p}\right)
\]

for all $x \in X$ and all $t > 0$.

**Proof.** If we define

\[
\mu_{(x_1, x_2, \ldots, x_p)}(t) = \frac{t}{t + \theta \sum_{i=1}^{n} ||x_i||^p}
\]

and apply Theorem 3, then we get the desired result.
Theorem 4. Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \) for which there is a \( \rho : X^n \to D^+ \) satisfying (2). If

\[
T_k^\infty \rho(2^{k+i-1}x,-2^{k+i-1}y,0,...,0) \left( 2^{k+2l-1}t \right) = 1
\]

and

\[
\lim_{m \to \infty} \rho(2^m x,2^m y,-2^m(x+y),0,...,0) \left( 2^{m+1}t \right) = 1
\]

for all \( x, y \in X \), all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \), then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\mu_{f(x)-Q(x)}(t) \geq T_k^\infty \rho(2^k x,0,...,0) \left( 2^{k-1}t \right)
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. Letting \( x_1 = x, x_2 = -x, x_3 = \ldots = x_n = 0 \) in (2), we get

\[
\mu_{2^i f(2x)-4^i f(x)}(t) \geq \rho(x,-x,0,...,0)(t)
\]

which is equivalent to

\[
\mu_{f(x)-\frac{1}{4} f(2x)} \left( \frac{t}{4} \right) \geq \rho(x,-x,0,...,0)(2t)
\]

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) and \( t \) by \( 2^{k-1}x \) and \( 2^{k-2}t \), respectively in the above inequality, we get

\[
\mu_{\frac{1}{4k} f(2^{k-1}x)-\frac{1}{4k} f(2^k x)} \left( \frac{t}{2^k} \right) \geq \rho(2^{k-1}x,-2^{k-1}x,0,...,0)(2^{k-1}t)
\]

for all \( x \in X \) and all \( t > 0 \).

Since \( \mu_x(s) \leq \mu_x(t) \) for all \( s \) and \( t \) with \( 0 < s \leq t \), we obtain

\[
\mu_{\frac{1}{4k} f(2^{k-1}x)-\frac{1}{4k} f(2^k x)} \left( \frac{t}{2^k} \right) \geq \mu_{\frac{1}{4k} f(2^{k-1}x)-\frac{1}{4k} f(2^k x)} \left( t \right)
\]

\[
\geq T_k^\infty \rho(2^{k-1}x,-2^{k-1}x,0,...,0) \left( 2^{k-1}t \right)
\]

Replacing \( x \) by \( 2^l x \) in the above inequality, we get

\[
\mu_{\frac{1}{4k} f(2^l x)-\frac{1}{4k} f(2^{m+l} x)} \left( t \right) \geq T_k^\infty \rho(2^{k+l-1}x,-2^{k+l-1}x,0,...,0) \left( 2^{k+l-1}t \right)
\]

which is equivalent to

\[
\mu_{\frac{1}{4k} f(2^l x)-\frac{1}{4k} f(2^{m+l} x)} \left( t \right) \geq T_k^\infty \rho(2^{k+l-1}x,-2^{k+l-1}x,0,...,0) \left( 2^{k+l+2l-1}t \right)
\]

for all \( x \in X \), all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \).
Since the right hand side of the inequality (18) tends to 1 as $m \to \infty$ by (15), the sequence \( \frac{1}{4m} f (2^m x) \) is a Cauchy sequence. Thus we define \( Q(x):= \lim_{m\to\infty} \frac{1}{4m} f (2^m x) \) for all \( x \in X \), which is an even mapping.

Now we show that \( Q \) is a quadratic mapping. By (2), we get
\[
\mu \frac{1}{4^m} (f(2^m(x-y)) + f(2^m(x+y)) + f(2^m(x+2y)) - 3f(2^m(x+y)) - 3f(2^m x) - 3f(2^m y)) (t) \\
\geq \rho (2^m x, 2^m y, -2^m (x+y), 0, \ldots, 0) (2^{m+1} t).
\]
Taking the limit as \( m \to \infty \) in the above inequality, by (16), the mapping \( Q \) is quadratic. Moreover, letting \( l = 0 \) and taking the limit as \( m \to \infty \) in (18), we get (17).

The rest of the proof is the same as in the proof of Theorem 3.

**Corollary 4.** Let \( \theta \geq 0 \) and let \( p \) be a constant with \( 0 < p < 2 \). For a normed vector space \( X \) and complete RN-space \( Y \), let \( f : X \to Y \) be an even mapping satisfying
\[
\mu_{Df(x_1, x_2, \ldots, x_n)} (t) \geq \frac{t}{\theta \sum_{i=1}^{n} ||x_i||^p}
\]
for all \( (x_1, x_2, \ldots, x_n) \in X \) with \( \sum_{i=1}^{n} x_i = 0 \) and all \( t > 0 \). If
\[
T^\infty_{k=1} \left( \frac{2^{k+2l-2} t}{2^{k+2l-2} t + 2^{k+l} \rho ||x||^p} \right) = 1
\]
for all \( x \in X \), all \( t > 0 \) and all \( l = 0, 1, 2, \ldots \), then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\mu_{f(x)-Q(x)} (t) \geq \lim_{m\to\infty} T^m_{k=1} \left( \frac{2^{k-2} t}{2^{k-2} t + 2^{k} \rho ||x||^p} \right)
\]
for all \( x \in X \) and all \( t > 0 \).

**Proof.** If we define
\[
\rho_{(x_1, x_2, \ldots, x_n)} (t) = \frac{t}{\theta \sum_{i=1}^{n} ||x_i||^p}
\]
and apply Theorem 4, then we get the desired result.

**4. Hyers-Ulam Stability of the Functional Equation (1)**

We note that if a mapping \( f : X \to Y \) satisfies the functional equation (1), then the mapping \( f \) is realized as the sum of an additive mapping and a quadratic mapping [see 2, Lemma 2.1].

Here, we let \( g(x) := \frac{1}{2} (f(x) - f(-x)) \) and \( h(x) := \frac{1}{2} (f(x) + f(-x)) \) for all \( x \in X \). Then \( g(x) \) is an odd mapping and \( h(x) \) is an even mapping satisfying \( f(x) = g(x) + h(x) \). Moreover, we get the following:
\[
Dg(x_1, x_2, \ldots, x_n) = \frac{1}{2} \{ Df(x_1, x_2, \ldots, x_n) - Df(-x_1, -x_2, \ldots, -x_n) \}.
\]
\[ Dh(x_1, x_2, \ldots, x_n) = \frac{1}{2} \{ Df(x_1, x_2, \ldots, x_n) + Df(-x_1, -x_2, \ldots, -x_n) \}\]

for all \( x_1, x_2, \ldots, x_n \in X \).

Note that \( Df(x_1, \ldots, x_n) = 0 \) implies that \( Dg(x_1, \ldots, x_n) = 0 \) and \( Dh(x_1, \ldots, x_n) = 0 \).

**Theorem 5.** Let \( f: X \to Y \) be a mapping with \( f(0) = 0 \) for which there is a \( \rho : X^n \to D^+ \) such that

\[
\mu_{Df(x_1, x_2, \ldots, x_n) + Df(-x_1, -x_2, \ldots, -x_n)}(2t) \geq \rho(x_1, x_2, \ldots, x_n)(t)
\]

\[
\mu_{Df(x_1, x_2, \ldots, x_n) - Df(-x_1, -x_2, \ldots, -x_n)}(2t) \geq \rho(x_1, x_2, \ldots, x_n)(t)
\]

for all \( x_1, x_2, \ldots, x_n \in X^n \) and all \( t > 0 \). If \( \rho \) satisfies (3), (11) and (12), then there exists an additive mapping \( A: X \to Y \) and a quadratic mapping \( Q: X \to Y \) such that

\[
\mu_{f(x) - A(x) - Q(x)}(2t) \geq T \left( T_{k=1}^{\infty} \rho \left( \frac{nt}{2^{2k-2}} \right), T_{k=1}^{\infty} \rho \left( \frac{t}{2^{3k-3}} \right) \right)
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Consider an odd mapping \( g(x) := \frac{1}{2}(f(x) - f(-x)) \) and an even mapping \( h(x) := \frac{1}{2}(f(x) + f(-x)) \) for all \( x \in X \) with \( f(x) = g(x) + h(x) \). By Theorem 1, there exists a unique additive mapping \( A: X \to Y \) such that

\[
\mu_{g(x) - A(x)}(t) \geq T_{k=1}^{\infty} \rho \left( \frac{nt}{2^{2k-3}} \right)
\]

for all \( x \in X \) and all \( t > 0 \). And by Theorem 3, there exists a unique quadratic mapping \( Q: X \to Y \) such that

\[
\mu_{h(x) - Q(x)}(t) \geq T_{k=1}^{\infty} \rho \left( \frac{t}{2^{3k-3}} \right)
\]

for all \( x \in X \) and all \( t > 0 \). Since \( f(x) = g(x) + h(x) \), we obtain

\[
\mu_{f(x) - A(x) - Q(x)}(2t) = \mu_{g(x) - A(x) + h(x) - Q(x)}(2t)
\]

\[
\geq T \left( \mu_{g(x) - A(x)}(t), \mu_{h(x) - Q(x)}(t) \right)
\]

\[
\geq T \left( T_{k=1}^{\infty} \rho \left( \frac{nt}{2^{2k-2}} \right), T_{k=1}^{\infty} \rho \left( \frac{t}{2^{3k-3}} \right) \right)
\]

for all \( x \in X \) and all \( t > 0 \), as desired.

Similarly, we can obtain the following. We will omit the proof.

**Theorem 6.** Let \( f: X \to Y \) be a mapping with \( f(0) = 0 \) for which there is a \( \rho : X^n \to D^+ \) satisfying (19) and (20). If \( \rho \) satisfies (7), (15) and (16), then there exists an additive mapping \( A: X \to Y \) and a quadratic mapping \( Q: X \to Y \) such that

\[
\mu_{f(x) - A(x) - Q(x)}(2t) \geq T \left( T_{k=1}^{\infty} \rho \left( 2^{k-1}x, 2^{k-2}x, -2^{k-1}x, 0, \ldots, 0 \right)(2nt), T_{k=1}^{\infty} \rho \left( 2^{k-1}x, 2^{k-2}x, 0, \ldots, 0 \right) \left( 2^{k-1}t \right) \right)
\]

for all \( x \in X \) and all \( t > 0 \).
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References


